The Structure of the Triple Articulation

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1 Introduction

An enterprise takes ***power-to-the-edge*** when it is prepared to respond to demand situations in ways that address each demand's particular context-of-use; i.e. it seeks to satisfy ***asymmetric demand***.

This contrasts with holding ***power-at-the-centre***, where there is a formulation of forms of demand that is common to all the demand situations that the enterprise is prepared to satisfy; i.e. it seeks to satisfy ***symmetric demand***.

Taking power-to-the-edge therefore presents the infrastructures of an enterprise with a particular challenge: are they ***agile*** enough to satisfy the variety of forms of demand encountered at its edges?

In terms of the structure described below, the agency of an enterprise that holds power-at-the-centre can be accounted for by the ***deontic*** and ***existential*** articulations alone, the deontic authorising just those behaviours that the enterprise is prepared to engage in.

To account for power-to-the-edge, however, we must introduce a ***referential*** articulation, representing a multiplicity of contexts-of-use encountered at the enterprise's edges. In this way, authority passes elsewhere, to that which is authorised by any particular context-of-use which the enterprise seeks to satisfy.

Triple articulation provides the means of evaluating the risks encountered in taking power-to-the-edge. It is at the heart of a praxis of knowledge elicitation and strategic intervention in enterprises that has been developed and practiced for 15 years by Boxer & Co. [http://www.brl.com/boxer_and_co.html].

The intuitions underlying triply articulated modelling are:

- that an *enterprise* is the subject of the experience of being *incorporated*;
- that an *actor* in an enterprise is the subject of the experience of being *embodied*;
- that an observer is a reflective actor, that is an actor able to articulate his or her experience as an embodied subject.
- that an observer may *articulate* its experience of:
  - being driven (the *referential articulation*),
  - having intentions (the *deontic articulation*) and
  - having capabilities for action and maintaining a history of having acted (the *existential articulation* and its related *trace articulation*, respectively); and
- that the formation of the observer-actor as a subject entails the *composition* of these articulations (a triple articulation).
- that an observer may describe an enterprise as the composition of its observer-actors' triple articulations.

Composition may be either

- *homogeneous*, involving articulations of the same type but by different actors; or
- *heterogeneous*, involving articulations of different type by the same actor.

Heterogeneous composition is not commutative, so the three articulations may be composed sequentially in six different ways. Furthermore, it is in the nature of the observer-actor's experience that not everything can be said about it and, therefore, that the observer-actor's formation as a triple articulation always has *gaps*. The nature of these gaps present particular challenges to a praxis of strategic intervention and are manifested:

- for the actor, in relation to the referential articulation as differences between need and demand, and to the deontic articulation between espoused theory and theory-in-use ; and
- for the enterprise, in relation to the composite deontic articulation as differences between formal and informal organisation, and to the existential between intended and emergent coupling.

Insofar as these challenges are overcome, the triply articulated model can be used as an adequate representation of the enterprise. The composition of the articulations in this model will exhibit 'holes' which connote *risks* to which the enterprise is exposed. These holes relate to the way in which the enterprise seeks to respond, sustainably, to the demands that it encounters; they represent, at worst, threats to its continuing viability and, at best, opportunities for its strategic development.

The structure and composition of articulations make these risks visible and quantifiable, while their hypothetical revision enables alternative strategies to be evaluated.

2 The Triple Articulation

2.1 The existential articulation

This structure models the actor's knowledge of how her world behaves in terms of:

- *processes*, which are (objectively) observable by the changes that they induce in the (public) state-of-affairs;
- *events*, which are states-of-affairs that are observed to pertain after the occurrence of a process; and
- *coordinations*, which are collections of processes (and/or coordinations, recursively) that are observed to occur together in some purposeful way.

This model gives a formal account of the world observed by the actor, explicating its material behaviour in terms of processes that act in it. In this kind of model, which when elaborated becomes a scientific theory, one event logically entails another if there is a process that is enabled by the first and terminates in the second. This formal model is related to the material world by an interpretation that takes each event to a state-of-affairs and each process to a 'material cause', in such a way that every chain of logical entailments commutes with an observably causal effect. The 'lower' one goes down the recursive existential articulation, the 'deeper' become the 'formal causes' in terms of which the material behaviour of the world is explicated. Of course, like all formal theories, the existential articulation may be inconsistent or erroneous, subject to alteration as the actor learns more about her world. To be complete, it would have to include a 'theory of everything', which is a paradoxical concept.

In the existential articulation, the actor models her world from an *exo* [ATM] perspective. Standing 'outside' of it, she can envisage its behaviour with its causality reversed, tracing effects back to their (formal) causes. But she can account for the effects on herself of processes in her world only by constructing a model of her world from the *endo* perspective (with herself in it) where, although she experiences as irreversible the occurrence of states-of-affairs, she may nevertheless know that the concurrency of certain of these occurrences may be reliably controlled.

2.2 The deontic articulation

This structure models the actor's knowledge of this controllability in terms of:

- *outcomes*, which are observable states-of-affairs;
- *transformations*, which denote, but do not necessarily give an account of, mechanisms that alter outcomes; and
- *synchronisations*, which are collections of outcomes (and/or synchronisations, recursively) that may be made to occur together.

The existential and deontic articulations may be composed by asserting mappings among their respective components that induce the implication of certain existential events in certain deontic outcomes. As with the existential, this **composed existential-deontic** (double) articulation may be inconsistent or erroneous. In addition, although the existential models each process as a 'closed system', that interacts only through the events to which it is directly related, not all of these interactions may be mapped into the deontic. Indeed, the existential's incompleteness leaves open the possibility of behavioural interactions that are not represented at all. These 'open system' phenomena, such as 'feature interaction' and 'emergent behaviour', are therefore necessarily lacking from the composed double articulation.

This double articulation defines the repertoire of all behaviour paths known to the actor. It is the space in which the actor, as 'efficient cause', can act by constructing and executing plans that make certain behaviour paths, and hence certain states-of-affairs, come to pass. Which plans the actor chooses will depend on how she values their implicated behaviour paths and outcomes. To give an account of her choosing, we must construct a model, not of the material world, but of the embodied individual as 'final cause'. This (*exo*) account is of the actor's anticipation of the effects in herself — as a particular 'other' — of experiencing certain behaviour paths, and of how her choice among these behaviour paths is entailed by that anticipation. This reversal of the direction of entailment (from 'efficient' to 'final' cause) characterises the 'anticipatory system' [ROS].

2.3 The referential articulations

This structure models the actor's knowledge of herself as an anticipatory system in terms of:

- *drivers*, which attribute value to the actor's experience (by being, more or less, 'satisfied' by paths-of-behaviour);
- *demand situations*, which are states-of-affairs whose coming to pass is anticipated to be satisfied with respect to certain drivers; and
- *customer situations*, in which the experience of situations (recursively) common to certain demand situations is anticipated to be satisfied with respect to certain drivers common to those demand situations.

The direction of 'higher' and 'lower' is reversed in the referential articulation, so that the 'higher' one goes up the referential articulation, the more of the actor's being is implicated in the demand situations, and the more complex becomes the actor's valuation of her experience of them in terms of drivers (and of their supporting behaviour paths). Equally, the 'lower' one goes down (away from the demand situations), the 'simpler' become the patterns of behaviour entailed by the actor's anticipation of their experience (and of their supporting behaviour paths), until the actor becomes indifferent to the way the situation is experienced *per se* (referred to as requirements, being context-independent customer situations).

As in the other two articulations, the referential may be erroneous or inconsistent and, since the actor's anticipation of her world always 'leaves something to be desired', is necessarily incomplete. In particular, this fundamental human limitation restricts the actor's ability to observe, and to express:

- her experience of being driven and
- the extent of contexts in which she anticipates satisfaction.

This form of the referential articulation, as elicited from the actor, does not provide the structure or content required for composition with the other two articulations. It is possible, however, to ***construct*** from the elicited data a ***hypothetical*** **pseudo-referential** articulation which provides the minimal required structure and content and is consistent with the elicited data.

This constructed pseudo-referential articulation may now be **composed** with the others by asserting mappings between events, requirements and outcomes, or between collections of drivers, coordinations of behaviour and synchronisations of outcome.

2.4 The composite triple articulation

The mappings asserted during composition may induce inconsistencies among the articulations. The exposure of these symptoms of invalidity to the actor, and her consequent repair of them by appropriately updating her models, accelerate her learning and facilitate her strategic development.

Given an internally consistent, **composite triple** articulation, and choosing any of its three component articulations as a basis, we may 'prune' one of the others by eliminating from it all elements which are not mapped to from the former. This pruning operator can then be repeated with the third articulation. Since it is not commutative, there are six different ways of applying it to a composite articulation, each of which reveals the ways in which assumptions in one articulation induce gaps in the others.

These six sequences of pruning are illustrated below:

synchronisations of outcome	⟶	coordinations of behaviour	≡	**role hierarchy**	⟶	anticipations of satisfaction	≡	**what don't we know how to satisfy?**
synchronisations of outcome	⟶	anticipations of satisfaction	≡	**supply-side goal stratification**	⟶	coordinations of behaviour	≡	**what can't we do from the supply-side**
anticipations of satisfaction	⟶	*synchronisations of outcome*	≡	**demand-side goal stratification**	⟶	coordinations of behaviour	≡	**what can't we do from the demand-side**
coordinations of behaviour	⟶	*synchronisations of outcome*	≡	**capability hierarchy**	⟶	anticipations of satisfaction	≡	**what are we unable to satisfy?**
coordinations of behaviour	⟶	anticipations of satisfaction	≡	**supply-side value stratification**	⟶	*synchronisations of outcome*	≡	**what are we unable to orchestrate?**
anticipations of satisfaction	⟶	coordinations of behaviour	≡	**demand-side value stratification**	⟶	*synchronisations of outcome*	≡	**what don't we know how to orchestrate?**

3 A Formal Model of Articulations[1]

An articulation is a triple comprising
Z, the **zero-level graph** of the articulation (a directed acyclic graph, see Appendix 1),
D, the **DAG** of the articulation, whose nodes we refer to as the **apices** of the articulation. All the leaves of D, but none of its apices, are vertices and edges of Z.
N, the set of names that label the objects (vertices, edges and apices) of each articulation with names that the actor assigns to corresponding objects in her specific world[2].
No two different objects of an articulation may have both the same type and the same name[3].
The **objects**, O, of an articulation are its apices, vertices and edges.

$$\mathbf{ART} = (Z, D, N)$$
$$\quad apices = D.nodes$$
$$D.leaves \subseteq (Z.V \cup Z.E) \wedge apices \cap (Z.V \cup Z.E) = \varnothing$$
$$N: (apices \cup Z.V \cup Z.E) \rightarrow String$$
$$\quad inj(N \uparrow apices) \wedge inj(N \uparrow Z.V) \wedge inj(N \uparrow Z.E)$$
$$O = apices \cup D.E \cup Z.V \cup Z.E$$

The **height** of an apex is the shortest distance between the apex and a leaf.
An apex is said to be **above** a leaf that defines its height.
The apices are partitioned into two classes:
PA, the **primary** apices and
DA, the **dual** apices,
subject to the constraint that **primary apices** are **above vertices** of Z and **dual apices** are **above edges** of Z.

$$height: apices \rightarrow NAT$$
$$\quad \forall a: apices.$$
$$\qquad height(a) = \min\{distance(a,b) \mid b \in D.leaves\}$$
$$above: apices \leftrightarrow D.leaves$$
$$\quad \forall a: apices;\ b: D.leaves.$$
$$\qquad above(a,b) \Leftrightarrow height(a) = distance(a,b)$$
$$\quad apices = PA \cup D$$
$$\quad PA \cap DA = \varnothing$$
$$\quad above \in (PA \leftrightarrow Z.V) \cup (DA \leftrightarrow Z.E)$$

The **primary subdag**, PS, is the subgraph of D whose nodes are the primary apices and whose leaves are vertices of Z.
The **dual subdag**, DS, is the subgraph of D whose nodes are the dual apices and whose leaves are edges of Z.
The **connecting subdag**, CS, is the subgraph of D whose edges, CS.E, connect the primary and dual apices.
The sets of edges of these subdags **partition** the edges of D and the **source and range functions of the edges of D** are likewise paritioned.

$$PS = vsub((PA \cup Z.V), D)$$
$$DS = vsub((DA \cup Z.E), D)$$
$$CS = esub((D.E - (PS.E \cup DS.E)), D)$$
$$\quad \{PS.E, DS.E, CS.E\} \in part(D)$$
$$PS.s = D.s \uparrow PS.E \qquad PS.r = D.r \uparrow PS.E$$
$$DS.s = D.s \uparrow DS.E \qquad DS.r = D.r \uparrow DS.E$$
$$CS.s = D.s \uparrow CS.E \qquad CS.r = D.r \uparrow CS.E$$

Each primary apex, a, **denotes** the subgraph of Z bounded by the vertices it subtends. This subgraph, Z_a, is equivalent to **the union of the subgraphs** denoted by its non-terminal descendants.

$$\forall a: PA.\ Z_a = vsub(PS.term[\{a\}], Z)$$

Each dual apex, a, **denotes** the set of edges of Z that it subtends. This set of edges is equivalent to **the union of the sets of edges** denoted by its non-terminal descendants.

$$\forall a: DA.\ E_a = DS.term[\{a\}]$$

3.1 Varieties of Articulation

In an **existential** articulation

- the vertices of the zero-level graph are *processes*
- the edges of the zero-level graph are *events*
- the primary apices of the dag are *coordinations* of processes and
- the dual apices of the dag are *views* of events.

1 The usual notation of set theory and first order predicate calculus is used throughout. See Appendix 2.
[2] The *types* of those objects ('event', 'process' etc.) and the names labelling the arcs between them (''supplies', 'derives', 'contains' etc.) are elements of PAN's generic metalanguage.
[3] Note that two objects of different type may have the same name.

In a **deontic** articulation

- the vertices of the zero-level graph are *outcomes*
- the edges of the zero-level graph are *transformations*
- the primary apices of the dag are *synchronisations* of outcomes and
- the dual apices of the dag are *compositions* of transformations.

Because of the fundamentally human limitations, that we noted earlier, on the actor's ability to observe, and to express, her experience of being driven and the extent of contexts in which she anticipates satisfaction, she cannot directly associate drivers with demand situations.

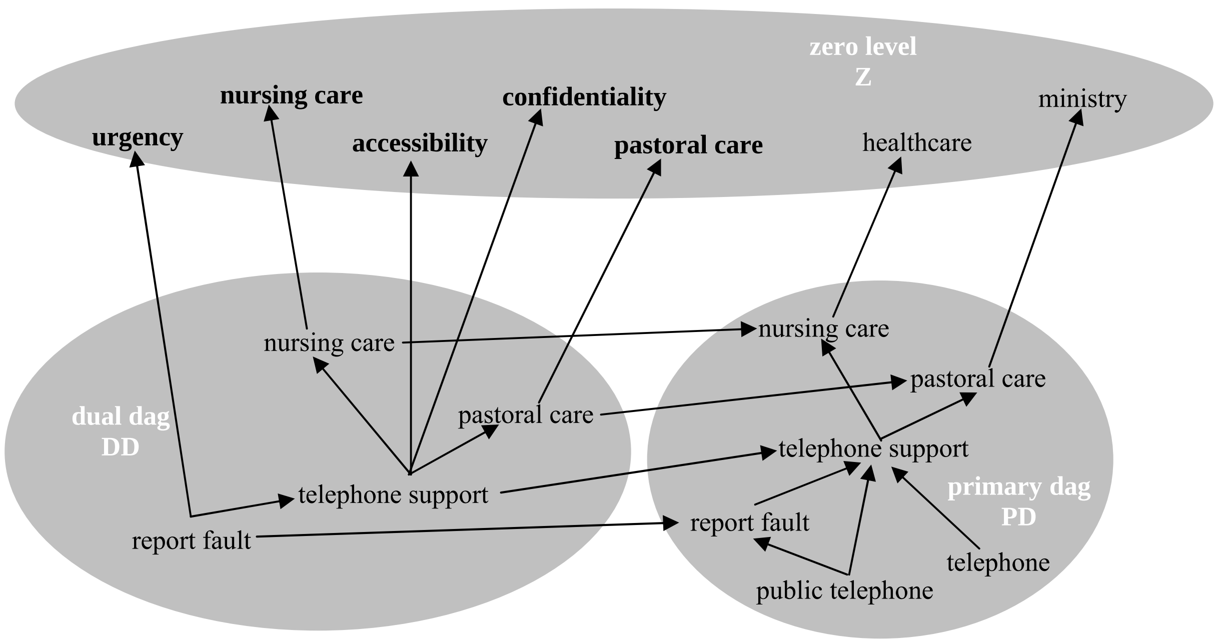
The zero-level graph of the elicited referential articulation is therefore degenerate in that its source and target functions are empty (that is, its vertices and edges are completely disconnected).

In a **referential** articulation**,** as elicited from the actor,

- the vertices of the zero-level graph are *demand situations*
- the edges of the zero-level graph are *drivers*
- the primary apices of the dag are *customer situations* and *requirements* and
- the dual apices of the dag are *sets* of drivers.

An example of an elicited referential articulation is shown in figure 1. Here, the drivers are in bold face, the demand situations are in plain, and those vertices of the primary dag that are driven are customer situations while those that are not are requirements.

Figure 1: An Observed Referential Articulation



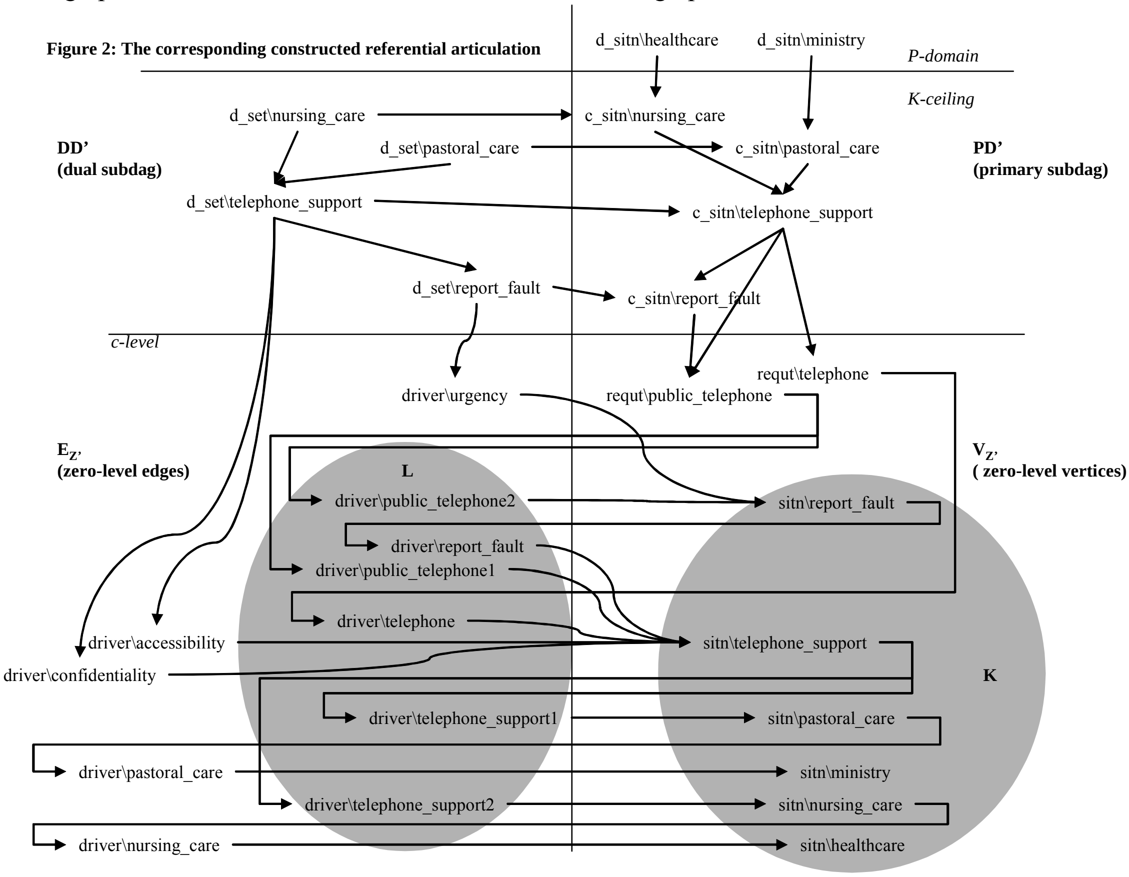
3.2 The Constructed Referential (or *Pseudo-Referential*)

In a **constructed pseudo-referential** articulation,

- the vertices of the zero-level graph are *requirements*
- the edges of the zero-level graph are *drivers*
- the primary apices of the dag are *customer situations*,
  each being a *context of use* in which the subtended requirements arise, and
- the dual apices of the dag are *value profiles*,
  each being an *anticipation of satisfaction* identified with the subtended set of drivers.

The constructed pseudo-referential articulation corresponding to the observed one in figure 1 is shown in figure 2. Note that the effect of this construction is to turn the dag upside down, so that what was further from the observed zero-level graph now become closer to the constructed zero-level graph.

Figure 2: The corresponding constructed referential articulation



We refer to such a constructed articulation as a *pseudo-referential articulation* because of its inverted relation to the referential combined with its constructed form. As in the deontic articulation, its vertices are 'event-like' and its edges 'process- like' objects. Henceforth, we will refer to elicited referential articulations as, simply, referential and to constructed referentials as pseudo-referentials.

3.3 Computing the Pseudo-referential

The transformation from a referential articulation, (Z, D, N), to the pseudo-referential, (Z', D', N'), is computed as follows:

3.3.1 Renaming

All the objects in the referential articulation retain their names in the corresponding pseudo-referential:

$N \subset N'$

3.3.2 Introduction of new components

Introduce sets K, L, so that disj{K, L, Z.V, Z.E, D.V, D.E}, and define surjective (one-to-one and onto) functions:

to generate a unique *situation* for each *customer-situation* and *demand-situation* in the primary subdag, D.PS;

k: (PS.V - PS.roots) $\rightarrow$ K

to introduce a unique *driver* for each edge[4] in the primary subdag;

l: PS.E $\rightarrow$ L

and to name them appropriately.

$\forall$x: (PS.V - PS.roots). N' k x = N x $\forall$y: (PS.E). N' l y = N y

4 Creating a new driver for each edge, instead of each node, in PS ensures that no edge in the constructed zero-level graph will have more than one vertex in its range.

3.3.3 Construction of the graphs

The vertices of the pseudo-referential zero-level graph are the *requirements* of the referential and the *situations* generated by k.

$$\mathbf{Z'.V} = \text{PS.roots} \cup \text{ran } k$$

The edges of the pseudo-referential zero-level graph are the *drivers* of the referential and those generated by l.

$$\mathbf{Z'.E} = \text{Z.E} \cup \text{ran } l$$

The source vertex of each edge in the pseudo-referential zero-level graph is given by:
if the edge was a driver in the referential, then it has no source vertex:
otherwise, the edge was generated (by l) from an edge in the primary subdag, in which case
if the source of that edge is a requirement,
then that is the required source vertex:
otherwise, the source of that edge is a customer situation, in which case the required vertex is the situation generated (by k) from that customer situation.

$$\text{Z.E}_Z \cap \text{dom } \mathbf{Z'.s} = \varnothing$$
$$\forall e: \text{PS.E. } (\text{PS.s}(e) \in \text{PS.roots} \rightarrow \mathbf{Z'.s}(l(e)) = \text{PS.s}(e)$$
$$\forall e: \text{PS.E. } (\text{PS.s}(e) \in (\text{PS.V - PS.roots}) \rightarrow \mathbf{Z'.s}(l(e)) = k(\text{PS.s}(e))$$

Note: the last two terms of may be combined in

$$\mathbf{Z'.s} = (k \cup \text{Id(PS.roots)}) \circ \text{PS.s} \circ l^{-1}$$

The range vertex of each edge in the pseudo-referential zero-level graph is given by:
if the edge was a driver in the referential, then its range vertex is the situation generated (by k) from the customer situation driven by the d-set of which that driver is a member:
otherwise, the edge was generated (by l) from an edge in the primary subdag, in which case the required vertex is the situation generated (by k) from the node at the source of that edge.

$$\forall e: \text{PS.E. } \mathbf{Z'}.r(l(e)) = k(\text{PS.s}(e))$$
$$\forall d:(\text{Z.E}). \ \mathbf{Z'}.r(d) = k(\text{CS.r } (\text{CS.s}^{-1} \ (\text{DS.s } (\text{DS.r}^{-1}(d)))))$$

Note: these two terms of may be combined in

$$\mathbf{Z'.r} = k \circ ((\text{PS.r} \circ l^{-1}) \cup (\text{CS.r} \circ \text{CS.s}^{-1} \circ \text{DS.s} \circ \text{CS.r}^{-1}))$$

The vertices of the pseudo-referential dag are just those of the referential dag and the edges of the pseudo-referential dag are just those of the referential dag,
but those of the primary dag (PS'.E) are reversed in direction.

$$\mathbf{D'.s} = \text{D.s} \dagger \text{PS.r}$$
$$\mathbf{D'.r} = \text{D.r} \dagger \text{PS.s}$$

$$\mathbf{D'.V} = \text{D.V}$$
$$\mathbf{D'.E} = \text{D.E}$$

3.3.4 The Twists

It will be noted that the vertices of the existential zero-level graph (processes) relate to the edges of the deontic zero-level graph (transformations) and vice versa, but direction of height of apices above the zero-level graph is the same in both articulations. This describes the **first twist.**
The relation of the observed referential to the deontic is that the vertices and edges in each relate to the vertices and edges in the other, but the direction of height above zero-level is reversed. This is the **second twist**. In these terms, the relation of the referential and existential articulations are characterised by both twists.
The effect of constructing a pseudo-referential is to remove the second twist, so that the relations of the pseudo-referential to the deontic retains the relationships of both the direction of height and of vertices and edges.

4 Composition

An articulation may contain a number of articulations, of the same type, made by different observers.

Homogeneous composition proceeds by asserting mappings and identities among the respective objects of such a collection of similar articulations, thereby constructing a single articulation of that type.

A composite articulation consists of one articulation of each type, with or without mappings amongs their respective objects.

Heterogeneous composition proceeds by selecting a pair of articulations, of different types, from the composite, selecting an object in one of them, identifying which objects in the other are candidates for mapping to the selected object, and choosing to effect one of those mappings.

In the definitions that follow, we consider a composite articulation of the form:

COMP = (**E**, **D**, **R**, **M**)

where **E**: N → **ART** is a named set of existential articulations
D: N → **ART** is a named set of deontic articulations,
R: N → **ART** is a named set of pseudo-referential articulations,
these names being distinct:
disj(dom **E**, dom **D**, dom **R**)

and **M**: **O** ↔**O**
where **O** =ran(**E** ∪ **D** ∪ **R**).O
is the set of all objects in the composite articulation,
is a relation in which each pair is a mapping between objects in different articulations:
∀(a,b):M. ∃x,y:N. x≠y ∧
(**E** ∪ **D** ∪ **R**)x =a ∧

The mappings asserted under heterogeneous composition are restricted by the following constraints:

4.1 Syntactic Constraints

ensure that the composite articulation satisfies the structural invariants defining the class of articulations, viz::

1. every apex must be *above* **exactly one** object of the zero-level graph;
2. the zero-level graph, the primary DAG and the dual DAG must all be **acyclic**; and
3. no two different objects of an articulation may have both the same type and the same name.

4.2 Semantic Constraints

define which classes of object may participate explicitly in the mappings, thus

event to *outcome*	zero-level, anti-clockwise
outcome to *requirement*	zero-level, anti-clockwise
composition to *coord'n*	apex-to-apex, clockwise
coordination to *d-set*	apex-to-apex, clockwise
c-sitn to *synchronisation*	apex-to-apex, clockwise
synchronisation to *view*	apex-to-apex, clockwise

∀C:**COMP**. C.**M** ⊆
(ran C.**E**).Z.E × (ran C.**D**).Z.V
∪ (ran C.**D**).Z.V × (ran C.**R**).Z.V
∪ (ran C.**D**).DA × (ran C.**E**).PA
∪ (ran C.**E**).PA × (ran C.**R**).DA
∪ (ran C.**R**).PA × (ran C.**D**).PA
∪ (ran C.**D**).PA× (ran C.**E**).DA

4.3 Candidates for Mapping

Given any two articulations, A and B, in a composite articulation and an apex, a, in A, an apex, b, in B, is a candidate for mapping with a if:

- b is a descendant of an apex in B to which a is mapped, or
- all the terminal descendants of a are terminal descendants of at least one non-terminal descendant of b.

explore: **COMP** → (**O** ↔**O)**
(a,b) ∈ explore(C)
↔ ∃A, B: (ran C.**E** ∪ ran C.**D** ∪ ran C.**R**).
a ∈ A.D.nodes ∧
b ∈ B.D.nodes ∧
∃c: B.D.nodes.
((c,b) ∈ nont ∧ (a,c) ∈ C.**M**)∨
((b,c) ∈ nont ∧ term{[a}] ⊆ term[{c}])

5 Towards an Algebra of Articulations

5.1 Introduction

The set-theoretic model presented above is derived from empirical practice. The structures and operations that it describes constitute a specification for the implemention, in Prolog, of the PAN system but, since that software was developed before this specification was constructed, it is yet to be verified.

Furthermore, the practice on which the model is based has itself been developed and elaborated during the construction both of the software and of the model.

We therefore seek to develop a more abstract formulation of articulations and of the operations that are performed on them, that would constitute a *theory of the practice*.

We expect this theory to illuminate the practice, enabling both the construction of experiments designed to seek its refutation and the design of tests designed to verify the code.

We also expect it to provide structural insights into the nature of embodiment under conditions of asymmetric demand that are cognate with those proposed but never successfully formalised by Freud [FRE]; and subsequently partially formalised by Lacan [e.g. in LAC72 pp57-58].

The first step in the construction of such a theory is the definition of an algebra whose objects are the graphs in which we present articulations.

Various properties of these graphs, particularly of the 'twists' to which they are subjected in the triple articulation, suggest that we should represent each articulation by the ***C*-algebra*** associated with its graph. A C*-algebra can be associated to a directed graph in such a way that the structure of the graph is reflected in the structure of the algebra [BRA72], [LAZ80], [RHO2001]. This representation would provide us with a powerful analytical basis for composition since the C*-algebra of a Cartesian product of graphs is a tensor product of the individual graph C*-algebras.

5.2 Mathematical Foundations

At the beginning of the 20th century, science saw a huge development in the application of concepts derived from structuralism. Structuralism is an approach that grew to become one of the most widely used methods of analyzing language, culture, philosophy of mathematics, and society in the second third of the 20th century. 'Structuralism', however, does not refer to a clearly defined 'school' of authors, although the work of Ferdinand de Saussure is generally considered a starting point. Structuralism is best seen as a general approach with many different variations. As with any cultural movement, the influences and developments are complex.

Broadly, structuralism seeks to explore the inter-relationships (the "structures") through which meaning is produced within an object. In Anthropology a structuralist focus was used by Lévi-Strauss, explaining that opposites are at the basis of social structure and culture.

In the early to mid 1960s, Noam Chomsky developed the idea that each sentence in a language has two levels of representation - a deep structure and a surface structure. The deep structure was a direct representation of the basic semantic relations underlying a sentence, and was mapped onto the surface structure (which followed the phonological form of the sentence very closely) via *transformations*. This was the origin of Chomsky's Transformational Generative Grammars (TGG). Chomsky emphasizes the importance of modern formal mathematical devices in the development of grammatical theory:

> "But the fundamental reason for [the] inadequacy of traditional grammars is a more technical one. Although it was well understood that linguistic processes are in some sense "creative", the technical devices for expressing a system of recursive processes were simply not available until much more recently. In fact, a real understanding of how a language can (in Humboldt's words) "make infinite use of finite means" has developed only within the last thirty years, in the course of studies in the foundations of mathematics." [CHO65]

A re-definition of structuralism has been proposed by many scholars, including U. Moulines and his group for whom:

> "Structuralism was born with Joe Sneed's and Wolfgang Stegmueller's work in the beginning of the 70's [5, 6]. It belongs to the non-statement view, a major stream of contemporary philosophy of science, which claims that the inner structure of a theory is more important than the statements, namely the laws, of the theory. That means also that the method to discover the laws is not of primary importance as long as we can organize them in a coherent structure. Such a theoretical structure is called a theory-net, which is a semi-lattice structure organizing laws of the theory…" [WCGM2004]

Structuralism was one of the key influences on Lacan's re-reading of Freud, which drew upon the insights of structuralism while going much beyond it in addressing the problematics of embodiment, identification and the subject's relation to the unconscious (see e.g. [LEA] for an account of the limitations of structuralism within the context of Lacan's *oeuvre*). Following the ideas of Lacan, particularly those relating to the formation of discourses [LAC70], an approach to understanding the nature of the enterprise has been developed [BOX], in support of which we have developed the following mathematical framework. We model each subject's articulations as directed graphs, whose elements are interpreted according to the articulation as described in Section 2. For example, in an existential articulation, the vertices of the graph refer to *atomic* event states and the edges refer to *atomic* processes; atomic in the sense that they are viewed as indecomposable with respect to some intermediate event. Paths in such a graph also represent processes, but those that have intermediate states and are built up from those which are atomic. Thus, a path should be interpreted as an *activity chain* which is punctuated by named event states and built up from atomic processes.

With this point of view, we introduce the fundamental graph, E_0, which represents all the events and processes that the observer-actor is able to refer to. We will sometimes call this graph the *context graph* of the enterprise or simply the context. It should be viewed as our named 'reality', describing the finest level of detail that we allow our existential articulation to refer to.

Thus, when we wish to describe the behaviour of an enterprise or a construct within the enterprise, we will do this in terms of the enterprise as modelled by the particular form taken by the composition of three different kinds of graphs, corresponding to the triple articulation.

Furthermore, we think that an adequate formalization for the triple articulation of an enterprise should be founded on a superstructure in the sense of [SL], or the ideas of [DC] related to Suppes predicates.

In this case, we will consider the previous formalizations made by Martino [CBM] where each of the articulations will be a graph. The idea behind our superstructure is to construct a cumulative hierarchy over a set of atoms. The main interest is to be able to locate in that universe the articulation in which we are interested. It is important to note, however, that we only consider this to be possible in relation to the existential, deontic and pseudo-referential articulations, although we expect the formalization to cast light on the nature of the relationship between the pseudo-referential and the referential itself.

We will construct a superstructure as in [AdF], but having a finite order and a finite collection of atoms or urelements. We will consider the Zermelo-Fraenkel axioms, without Regularity, including the Axiom of Choice and Atoms (abbreviated as ZF-AC). Since the collection of atoms is finite and the superstructure is finite, this set of axioms has no unpleasant consequences[5]

5.3 Construction of a Semantic Universe.

These syntactic structures are described as directed graphs (see Apendix A1). Given the nature of our problem — modelling observer-actor's models of enterprises — it is necessary to provide a semantic structure where those syntactic structures could be interpreted. That interpretation is equivalent to provide a semantic structure where the syntactic structures could be embedded. We defined, based on ideas from Da Costa, De Freitas, Sebastião e Silva and Stegmüller, a set-theoretical Universe with atoms (*urelements*) where those articulations "live".

5.3.1 Superstructure

Definition 1: We define the following "cumulative hierarchy", comprising a sequence of 'etages', V_i, $i \in \omega$.

$V_0(U) = U$

$V_1(U) = \wp(U) \cup U$

.

.

.

$V_n(U) = \wp(V_{n-1}(U)) \cup V_{n-1}(U)$.

Definition 2: (Kuratowski, 1921): The set $\{\{e^1\},\{e^1, e^2\}\}$ is called the *ordered pair* (e^1,e^2).

We define the *first coordinate* of an ordered pair (e^1, e^2) to be $\cup\cap(e^1, e^2)$.

The *second coordinate* of an ordered pair (e^1, e^2) will be $\cup(\cup(e^1, e^2)) - (\cap(e^1, e^2))$.

Note that in general we could define ordered triples and n-tuples, using a recursive definition:

If (e^1, e^2) is an ordered pair, then the ordered triple (e^1, e^2, e^3) is defined as the set $(e^1, (e^2, e^3))$.

In general the n-tuple $(e^1, e^2,\ldots, e^n)$ is defined by $(e^1, (e^2,\ldots, e^n))$.

[5] It is known that ZFA is inconsistent relative to ZFC, among its consequences being the possibility of constructing inside ZFA an infinite dimensional vector space without base. Cf. for example, Jech, T., *Set Theory*, Princeton, 1978.

Definition 3: Let A and B be sets. The *cartesian product* of A and B is the set $A\times B = \{(a, b): a\in A \wedge b\in B\}$.

Definition 4: A set $R\subseteq A\times B$ is called a *binary relation* between A and B.

If R is such that if $(e,e^1)\in R \wedge (e,e^2)\in R$, then $e^1 = e^2$, then we will say that R is a function

In that case, we prefer to use the notation f instead of R.

With these definitions of superstructure and relation, we can determine, for any directed graph G_0, in which étage of the superstructure it will be formed. Observe that due to the fact that the superstructure is a cumulative hierarchy, it is enough to determine in which étage a particular set will be formed, because it will also be in all the following étages.

Note that the set V of vertices belongs, by definition, to the étage 0, i.e., V_0.

As the set of edges E is a set of ordered pairs of vertices we can calculate in which étage it will be formed, thus:

Let $e = (v^1, v^2)$ be an edge.

By definition $(v^1, v^2)=\{\{v^1\},\{v^1, v^2\}\}$.

The singleton $\{v^1\}$ and the unordered pair $\{v^1, v^2\}$ both belong to the power set of U, and therefore to $V_1(U)$.

The edge $e\in \wp(V_1(U)) \subseteq V_2(U)$.

Now we know that the edge e is formed in $V_2(U)$, and will be found in any étage after $V_2(U)$.

Then the set of edges E will be formed in $V_3(U)$.

Let us determine the étage where the functions r and s are formed.

We have determined that the ordered pairs of vertices are formed in $V_2(U)$,

and because a function is no more than a set of ordered pairs, then it will be formed in $V_3(U)$.

Then a directed graph is totally formed at $V_3(U)$, and so are its sub-graphs.

Definition 5: An *articulation, A*, is a tuple,

$(A_{0,0}, A_{1,1}, \ldots, A_{1,r_1}, A_{2,1}, \ldots, A_{2,r_2}, A_{3,1}, \ldots, A_{i,j}, \ldots, A_{n,1}, A_{n,r_n})$

where $A_{0,0}$ is a directed graph, each $A_{1,k}$ is a sub-graph of $A_{0,0}$ and in general

for $i>1$, $A_{i+1,j}$ is a subset of $\{A_{i,1}, \ldots, A_{i,r_i}\}$.

At this point we want to see at which étage the components of an articulation will be formed.

As we have seen, the "initial" graph, $A_{0,0}$, is formed in $V_3(U)$, as is $A_{1,j}$, $2\leq j\leq r_1$.

As we define $A_{2,j} = \{A_{1,1}, \ldots, A_{1,r_1}\}$, then it will be formed in $V_4(U)$

and $A_{3,j} = \{A_{2,1}, \ldots, A_{2,r_2}\}$ in $V_5(U)$

and, in general, $A_{n,j} = \{A_{n,1}, \ldots, A_{n,r_n}\}$ will be formed in $V_{n+2}(U)$.

Definition 6: The i^{th} *level* of an articulation, A, is defined to be $A^i = \{A_{i,1}, A_{i,2}, \ldots, A_{i,r_i}\}$.

The i^{th} level contains r_i elements or constituents.

As i increases we say that we move to a higher level and as i decreases to a lower one[6].

Each construct, $A_{i,j}$, defined above has a double subscript with each element a natural number. The first indicates the level at which the construct occurs and hence the ordering derived from the natural numbers has some meaning. For the second subscript, we are also using the set of natural numbers, but their order has no meaning, since we have sets, not tuples. The levels are therefore completely formed in the étages as described above.

We intend the $A_{i,j}$ to be interpreted as *named* constructs within the observer-actor's model of the enterprise, built up from constructs from a level of complexity directly below. An observer-actor's model of an enterprise is expressed in terms of three articulations: existential, deontic and referential. They formalize what an enterprise 'can do', 'intends to happen', and 'anticipates satisfying' respectively. For example, in the existential articulation, the directed graph A_0 represents the possible events and processes to which we are able to refer. Each sub-graph in the first level therefore represents a named *co-ordination of behaviour* and the levels in each articulation describe co-ordinations of behaviour, synchronisations of outcome and anticipations of satisfaction of demand, respectively, that model the enterprise. Each $A_{i,j}$ in the higher levels represents a construct within the enterprise that co-ordinates, synchronizes or anticipates the satisfaction of the constructs on the level below.

Thus with this formalization, the levels above the first are co-ordinations of co-ordinations (C^2), synchronisations of synchronisations (S^2), or anticipations of anticipations (A^2).

We should stress that these constructs, $A_{i,j}$, are *not* all sets of sub-graphs, as it is evident from the construction of the superstructure. Their meaning is best understood through the $\overline{A_{i,j}}$ graphs [AM] which describe behaviour, although this ignores the way in which they are built up from the constructs at lower levels.

[6] These levels should not be confused with the *stratification* of the enterprise although both describe increasing levels of complexity. The term 'stratification' is used to refer to the characteristics of a projection of a triple composite articulation with respect to a given demand situation, which, given the nature of the second twist, takes the form of a number of stratified simplicial complexes.

Suppose that one were to model an enterprise in this fashion.

One may have a person in the enterprise who is able to repair all the different types of machine within the enterprise. One might represent this man in the first level of the existential articulation as the sub-graph of A_0 consisting of every edge in A_0 that is the repairing of a machine (and perhaps also the paths in A_0 that correspond to such repairs). At the next level we may want to collect people into workgroups, which utilise all the skills of the people within them and so are represented as subgraphs of the level below.

Note that our point of view is that these constructs are indecomposable in the following sense: the repair-man who forms part of a workgroup might only be used to repair certain types of machine but he retains the ability to repair all the machines in the company and so does the workgroup. If one wanted to represent the fact that the workgroup used the repair-man in only one capacity one would make changes at the lower level. That is, instead of having a single graph representing the repair-man's capabilities one would have multiple graphs corresponding to different uses of his skills.

Example: Suppose for a particular enterprise the graph $A_{0,0}$ = {{a,b,a}, {(a,b), (b,c), (a,c)}}. Then the level 1 of the articulation constructed over this graph will be formed of all the sub-graphs of $A_{0,0}$, viz:

$A_{1,1} = A_{0,0}$
$A_{1,2}$ = {{a,b}, {(a,b)}}
$A_{1,3}$ = {{b,c}, {(b,c)}}
$A_{1,4}$ = {{a,c}, {(a,c)}
$A_{1,5}$ = {{a,b,c}, {(a,b), (b,c)}
$A_{1,6}$ = {{a,b,c}, {(b,c), (a,c)}
$A_{1,7}$ = {{a,b,c}, {(a,b), (a,c)}
$A_{1,8} = \varnothing$

Here, level 1 is formed by 8 elements.

One of the elements of level 2, say, $A_{2,j} = \{A_{1,2}\ ,\ A_{1,6}\ ,\ A_{1,7}\}$. Note that the cardinality of each level grows exponentially: in level 2, we will have 248 elements. In general, level *i* will have $|\ A_{i,j}\ | = 2^{r_i} - r_i$ elements.

5.3.2 Homogenous Composition

In order to construct a composite triple articulation for an enterprise, of which two observer's models have been presented, we must first construct the superstructure supporting their composition. An observer's model of an enterprise is modelled through the composition of the three types of articulation by that observer. Approaching composition within the context of such a composite triple articulation is referred to as *top-down* modelling of the observer's model. Where individual articulations of a particular type are composed before being composed with articulations of a different type, this is referred to as *bottom-up* modelling.

Since the set of atoms of the new superstructure is formed by taking component atoms, the new universe will be larger and richer than each of the others.

In order to be able to identify, for any vertex in a composite articulation, the component articulation to which it belonged, we will use a disjoint union for the sets of atoms.

Suppose that we have two existential articulations
$E = (E_0, E_{1,1}, \ldots, E_{1,r_1}, \ldots, E_{n,1}, E_{n,r_n})$ and $F = (F_0, F_{1,1}, \ldots, F_{1,s_1}, \ldots, F_{m,1}, F_{m,s_m})$

Let U_E and U_F be the sets of atoms supporting the superstructures for each articulation respectively.

We define a disjoint union of those sets as follows:

$U_E \times \{0\} \cup U_F \times \{1\}$ (1)

Definition 7: Let U_{EF} be the set defined in (1). Then the *composition superstructure* will be

$V_0\ (U_{EF}) = U_{EF}$
$V_1\ (U_{EF}) = \wp(\ U_{EF}) \cup U_{EF}$
.
.
$V_n\ (U_{EF}) = \wp(\ V_{n-1}\ (U_{EF})) \cup V_{n-1}\ (U_{EF})$

This composition superstructure defines a common context within which articulations of the same type may be composed[7], creating

a *sub-atomic* level, which relates the atoms of the deontic and referential articulations to those of the existential, and a *hypothetical* relationship between the étages supporting the observed referential intersection dag and those of the other two articulations.

---

[7] A key reason why the 'twists' are needed to define the way atoms and/or dags in each articulation relate to the other two articulations is in order to define how these common contexts relate to each other.

Note that in $V_n(U_{EF})$, we can define graphs that are isomorphic to E_0 and to F_0. Then to each vertex $e_1 \in U_E \cap U_F$ there will be two compound vertices $(e_1,0)$ and $(e_1,1)$. The only difference between E_0 and F_0 and their respective isomorphic graphs inside $V_n(U_{EF})$ is that each vertex in the composition superstructure is an ordered pair instead of an atom.

5.3.3 Local Universes

Note that, according to our definition, an atom has the property of having no elements but being different from the empty set. The atoms of the composition superstructure are therefore not 'real' atoms. We solve this theoretical problem by defining a weaker form of atoms, that we will call *molecules*, by 'lifting' atoms, as follows:

Definition 8: Let e be an atom, then:

1. the set (e,n), where n is a natural number, is a molecule.
2. If e,f are atoms, then (e,f) is a molecule.
3. If p is a molecule, then (p,n), where n is a natural number, is a molecule.

Molecules are formed outside the superstructure, in a structure that we will call a *Local Universe*.

Definition 9: Let A be a set of atoms, such that if $V_n(U)$ is any superstructure, then $U \subset A$.

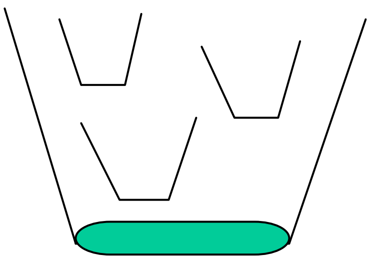
We define the Local Universe as a cumulative infinite hierarchy thus:

$$M_0(A) = A$$
$$M_1(A) = \wp(A) \cup A$$
$$\vdots$$
$$M_n(A) = \wp(M_{n-1}(A)) \cup M_{n-1}(A)$$
$$\vdots$$
$$M_\omega = \bigcup_{n=0}^{\omega} M_n(A)$$

Clearly any superstructure could be formed inside that one. It is important to note that there could exist many different, mutually incommensurable Local Universes.

By abuse of language, we will call each molecule an atom in any context where there is no risk of confusion.

We want to be able to form the composition of two articulations of a given type attributed to different observer-actors' models of the enterprise that describe the things that it can do. However it should be noted that the components will not themselves include any description of a co-ordination between the two different articulations since such a thing needs to be specified by the observer-actors and cannot be deduced. In fact it is perfectly possible that such a co-ordination does not exist.



In this figure, if we think that the three small superstructures inside of the Local Universe are the 'house' of three articulations of the same kind, then the subatomic level will be in the atomic level of the Local Universe, as it is the collection of all the atoms of the super-structures living inside that Local Universe.

The composition of two articulations of the same type is easy to define when the graphs of the respective articulations are isomorphic. It is only necessary to take as context one of the graphs and, for each level, form new levels that will be the union of the corresponding levels.

This definition makes two restrictions that we wish to avoid in general. Firstly it assumes that they only talk about the same things. In fact, we wish to be able to enlarge the language of each articulation so that they may at least have the ability to refer to the same things. Secondly, we wish to be able to compose articulations that have different numbers of levels. We achieve this by introducing extra levels that carry no more information[8].

It is important to note the following observations about molecules:

A molecule is an n-tuple of atoms.

As molecules are generated by the application of composition, they live inside the Local Universe, but we know only that they are formed from atoms, not where they live.

[8] Note that we are developing here an *Ontology* in the sense of Lesniewski [LES], as we are defining objects in a particular Universe using a semantical approach.

We know that they will determine the superstructure that the composite articulation will live in, because those molecules will form the set of *urelements* (atoms) that will generate it[9].

Definition 10: Let $E = (E_0, E_{1,1}, \ldots, E_{1,r_1}, \ldots , E_{n,1}, E_{n,r_n})$ and $F = (F_0, F_{1,1}, \ldots, F_{1,s_1}, \ldots , F_{m,1}, F_{m,s_m})$ be two articulations of the same type for two enterprises.

We define the *composition* of E and F as follows:

If E_0 is isomorphic to F_0 and $n = m$, so that they have the same number of levels,

then the composition of E and F is the articulation whose 0^{th} level, or *context graph*, is $E_0 \cong F_0$ and every level of the composite articulation is simply a union of the levels for the individual articulations.

In other words, the first level for the composite articulation is $E_{1,1}, \ldots, E_{1,r_1}, F_{1,1}, \ldots, F_{1,s_1}$.

In general the i^{th} level is $E_{i,1}, \ldots, E_{i,r_i}, F_{i,1}, \ldots, F_{i,s_i}$.

If $n < m$, then we add a level to E with exactly r_n parts, each of which is a singleton set containing a distinct member of the level below. Thus we have an articulation which carries the same information as E but which has one extra level.

Proceeding in this fashion we may assume that E and F have the same number of levels.

The composition graph, i.e. the common context of the composition articulation, will be defined inside the composition superstructure, using the disjoint union.

Thus to compose two articulations of the same kind E and F we add redundant levels to ensure that they have the same number of levels and we change the context graph, via some identified common sub-graph, so that they have the same graph at the bottom. That adds dummy levels to E, to express the fact that the questions that F asks in its articulation are wider than those addressed by E. E, however, does not grind to a halt in the face of these issues but replies by saying that the structures in place to deal with narrower issues will suffice for the broader issues also.

If we want to decompose a composite articulation of a given type, we need to look at its molecules which contain all the information about the graphs of each of its component articulations.

If the vertices of the graph are edges or apices, they are molecules in some superstructure.

5.4 Operations with structures.

Based on ideas in [MEN], we define operations for pruning an expansion in each articulation as a syntactic structure, and for concatenating them. These provide the mechanisms we need for the composition of articulations.

5.4.1 Refinement

Let E1 and E2 be (fragments of) existential articulations, where x, y and z are processes in E1; a, b, c and d are processes in E2; and A1 and A2 are co-ordinations that subtend sub-graphs in E2.

If the user identifies the vertex x in E1 with the apex A1 in E2, the identification is applied to the **union** of the graphs, generating a *refinement* of the *vertex* x, in which:

every edge **entering** x in E1 is made to enter every **originating vertex** in A1, and

every edge **leaving** x in E1 is made to leave every **teminating vertex** in A1.

If the user subsequently identifies the process y in E1 with the coordination A2 in E2, an inconsistency would be reported, because y would have to be identified with a sub-graph that contains it.

In the following definition of refinement, we identify the vertices of a graph with their sub-indices.

Definition 11: Let A_0 be a graph, with edges e_{ij} and vertices v_j.

Let M and N be subgraphs of A_0 where

$M = (V_M, E_M)$;

$V_M = \{i, i+1, i+2, ..., j: j>i\}$;

$E_M = \{(i, i+1), (i+1, i+2), ..., (j-1, j)\}$;

$N = (V_N, E_N)$;

$V_N = \{i, i+1, i+2, ..., k: k>i\}$;

$E_M = \{(i,l), ..., (m,i): i<l; i<m\}$

[Note that M must be a chain and that N must not be a chain.]

We define $Q = (M, N)$

[9] Henceforth, the word atom will refer to the ‘real’ atoms, 'molecule' to the former pseudo-atoms.and 'urelement' to either an atom or a molecule.

Then, given $P = (X, R)$, we define

$Q \otimes P = (Y, S)$

where $Y = X \times V_M$

and $S = \{((x,h), (x,h\text{-}1)) \mid i{+}1 \le h \le k \wedge x \in X\}$

$\cup \{((x,i), (x,k)) \mid x \in X\}$

$\cup \{((r,h), (r,h{+}1)) \mid i{+}1 \le h \le k \wedge r = (x,y) \wedge x \in X\}$

$\cup \{((x,i), (r,i{+}1)) \mid r = (x,y) \wedge x \in X\}$

$\cup \{((r,j\text{-}1), (y,i)) \mid r = (x,y) \wedge x \in X\}$

5.4.2 Heterogeneous Composition

Definition 12: Let E be an existential articulation and R a referential articulation. We define their composition, $E \bullet R$, as the articulation Δ defined by the base graph $\delta_0 = E_0 \otimes R_0$, where E_0, R_0 are the respective base graphs of E and R.

Definition 13: Let Δ be a composed articulation of an existential articulation E and R a referential articulation. Let Λ be a deontic articulation with base graph λ_0, then the composition $\Delta \lozenge \Lambda = \aleph$ is the articulation defined by the base graph, $gl(\delta_0, \lambda_0; \Pi)$, where Π is the equivalence relation defined for the operation 'gluing' (see below).

Example

Let $N_v = (\{1,2,3,4\}, \{4,3), (3,2), (2,1), (4,1)\})$ and

$N_e = (\{4,5,6,7,8\}, \{4,5), (5,6), (6,7), (7,8)\})$

Our intention is define a new graph replacing each vertex of some graph P by a copy of N_v and each edge of P by a copy of N_e, identifying the vertex 4 in each copy with the initial and final vertex of the chain N_e, thus:

Let $P = (X, R)$ and define

$N \otimes P = (Y, S)$, where

$Y = (X \times \{1,2,3,4\}) \cup (R \times \{5,6,7\})$

and the following couples are in the relation S:

(1) $((x,i), (x, i\text{-}1)) \in S$, for $i = 2,3,4$ and for $x \in X$;

(2) $((x,4), (x,1)) \in S$, for all $x \in X$;

(3) $((r,i), (x, i{+}1)) \in S$, for $i = 5,6$ and for all $r \in R$;

(4) $((x,4), (r, 5)) \in S$, whenever $r = (x,y)$, for all $x,y \in X$; and

(5) $((r,7), (y, 4)) \in S$, whenever $r = (x,y)$, for all $x,y \in X$.

If P is the graph $(\{a,b,c\}, \{(a,b), (b,c), (c,a)\})$, then the set of vertices Y in $N \otimes P$ will be

$\{(a,1), (a,2), (a,3), (a,4), (b,1), (b,2), (b,3), (b,4), (c,1), (c,2), (c,3), (c,4)\} \cup$

$(((a,b),5), ((a,b),6), ((a,b),7), ((b,c),5), ((b,c),6), ((b,c),7), ((c,a),5), ((c.a),6), ((c,a),7)\}$

Note that all the vertices of this new structure are molecules if the initial vertices are atoms.

5.4.3 Gluing graphs

Definition 14: Let $G_i = (X_i, R_i)$, $i \in I$, be a family of graphs, indexed by I.

Let E be an equivalence relation on $\bigcup_{i \in I} X_i$

Then we define a graph

$gl(G_i, i \in I; E) = (Z,T)$

where Z is the set of all equivalence classes of E and if z_1, z_2 are two such equivalence classes,

then $(z_1, z_2) \in T$ if and only if there exist $i \in I$ and $x_1, x_2 \in X_i$ such that $x_1 \in z_1$, $x_2 \in z_2$, and $(x_1, x_2) \in R_i$.

Example:

Let L be the family of graphs (K, C) where

$K = (\{1,2,3,4,5\}, \{(2,3), (3,4), 4,5), (5,1), (2,1)\}$,

$C = (\{6,7\}, \{(6,7), (7,6)\})$,

and the equivalence relation, E, on the set $\{1,2,3,4,5,6,7\}$ is defined by

$x \equiv y$ if and only if either $x, y \in \{2,3,4,5,6\}$ and $x = y$ or $x = 1$, $y = 7$ or $x = 7$, $y = 1$.

Then $gl(L, E) = (Z,T)$, where

$Z = \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1,7\}\}$and

$T = \{(\{6\},\{1,7\}), (\{1,7\},\{6\}), (\{2\},\{1,7\}), (\{2\},\{3\}), (\{3\},\{4\}), (\{4\},\{5\}), (\{5\},\{1,7\})\}$.

5.5 Order

Having constructed the semantic structures, where the set of atoms will represent the vertices of the articulations, we need to make that set explicit and provide it with an ordering relation.

In order to associate a C*-Algebra with each graph (following Bratteli [BRA72]), we could choose **C,** the set of complex numbers, to be the set of atoms. However, to avoid problems related to the cardinality of the continuum we will take a numerable subset of **C**, specifically the set of complex numbers whose real and imaginary parts are integers.

5.5.1 Lexicographic order

In order to deal with the association of complex numbers to vertices in a systematic way, we need to introduce a lexicographic order. The lexicographic order is the order of the dictionary applied to n-tuples belonging to a cartesian product of sets. In order to do that we assume that all tuples have the same length, if not we just complete them with 'dummy' characters.

Definition 15: Suppose $\{A_1, A_2, \ldots, A_n\}$ is a collection of sets, with respective total orderings $\{<_1, <_2, \ldots, <_n\}$.

Then the *lexicographic order*, $<^d$, on $A_1 \times A \times \ldots \times A_n$ is defined by

$(a_1, a_2, \ldots, a_n) <^d (b_1, b_2, \ldots, b_n) \Leftrightarrow \exists m{>}0.\ \forall i{<}m.\ (a_i = b_i) \wedge (a_m <_m b_m)$,

that is, if one of the terms $a_m <_m b_m$ and all the preceding terms are equal.

This is the ordering used in dictionaries, the words being read from left to right.

Similarly, *the Hebrew lexicographic order*, $<^h$, on $A_1 \times A \times \ldots \times A_n$ is defined by

$(a_1, a_2, \ldots, a_n) <^h (b_1, b_2, \ldots, b_n) \Leftrightarrow \exists m{>}0.\ \forall i{>}m.\ (a_i = b_i) \wedge (a_m <_m b_m)$.

This is the same ordering as before, but reading from right to left.

5.5.2 Labelling the elements of articulations

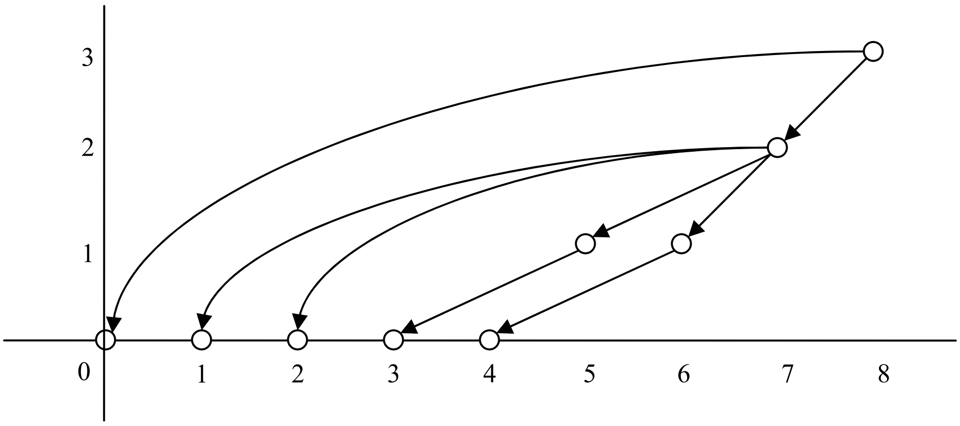
We attribute the complex numbers to the elements of an articulation in three steps:

1. identify the levels and tabulate their elements;
2. index the elements, attributing a different natural number to each;
3. to each element, attribute the complex number whose real part is the element's index and whose imaginary part is its level.

For example, applying this attribution to the dual dag, DS, of the elicited referential articulation shown in Fig. 1, we have:

Level 0:	driver/urgency	0+0i
	driver/accessibility	1+0i
	driver/confidentiality	2+0i
	driver/nursing_care	3+0i
	driver/pastoral_care	4+0i
Level 1:	dset/ nursing_care	5+1i
	dset/ pastoral_care	6+1i
Level 2:	dset/telephone_support	7+2i
Level 3:	dset/report_fault	8+3i

Re-drawing that dag in the complex plane, we have:



In the following, we use the the set of natural numbers defined as the smallest set (in the order of subsets) that contains 0 (the empty set) and is closed under the operation of successor.

Let V be the set of vertices and l-1 the number of levels in the dag, then $V = \bigcup_{0 \le b \le l-1} S_b$,

where S_b is the set of vertices in level b of the dag.

We define the following sequence of sets.

$$R_0 = |S_0|$$

$$R_1 = suc^{|S_1|}(|S_0|) - R_0$$

$$R_2 = suc^{|S_1|+|S_2|}(|S_0|) - R_1$$

$$R_3 = suc^{|S_1|+|S_2|+|S_3|}(|S_0|) - R_2$$

$$\vdots$$

$$R_l = suc^{\sum_{i=1}^{l}|S_i|}(|S_0|) - R_{l-1}$$

In the example above, this produces:

$R_0 = 3 = \{0, 1, 2\}$

$R_1 = suc^2(3) - 3 = 5 - 3 = \{0, 1, 2, 3, 4\} - \{0, 1, 2\} = \{3, 4\}$

etc.

Now define arbitrary surjective functions

$f_b : S_b \to R_b$, for $0 \le b < l$

and set

$$f = \bigcup_{i=0}^{l-1} f_i$$

Now define the following function,

$h : V \to \aleph_0$, where $\aleph_0$ represents the set of natural numbers.

$h(v) = b$ iff $v \in S_b$

Let g be the function defined by,

$g: V \to C$, where C is the field of the complex numbers,

$g = (f(v), h(v))$

$= f(v) + bi$

Now define the ordered set $A = \langle g(V), H \rangle$, where H is the Hebrew lexicographic order.

This is the set of all pairs of vertex labels, the first of which *follows* the second in lexicographic order.

If the original dag, D, was defined by the tuple $\langle V, E, r, s \rangle$, then the set of ordered pairs of vertex labels in the D is given by

$D_f = f \circ s \circ r^{-1} \circ f^{-1}$

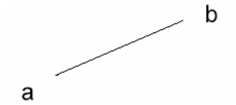
and the ordered set

$A \cap D_f$

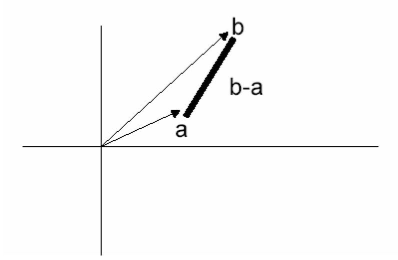
is isomorphic to D.

5.6 The C*-algebra of a graph

A path, P_n, is defined by a set of vertices $P_n^0 = \{0,1, 2, \ldots n\}$ and a set of edges $P_n^1 = \{e_0, e_1, \ldots, e_{n-1}\}$, for n>0 with $s(e_k) = k$ and $r(e_k) = k+1$.
Consider the graph P_1, where $P_1^0 = \{a, b)$, $P_1^1 = \{e\}$, s(e) = a and r(e) = b.



We insert that graph into the complex plane, thus:

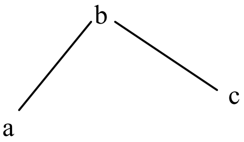


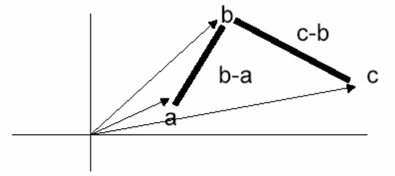
Then the graph P_1 is isomorphic to the complex number b-a, where $a = a_1 + a_2 i$ and $b = b_1 + b_2 i$.

Then $a = \begin{bmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{bmatrix}$, $b = \begin{bmatrix} b_1 & -b_2 \\ b_2 & b_1 \end{bmatrix}$ and the matrix representation for the graph P_1 is

$$b - a = \begin{bmatrix} b_1 - a_1 & -(b_2 - a_2) \\ b_2 - a_2 & b_1 - a_1 \end{bmatrix}$$

As we can see, that graph is isomorphic to a 2×2 real matrix, that is, a complex number, i.e., $C^*(P_1) \approx \boldsymbol{C}$ (see Appendix A3).
Now consider P_2.

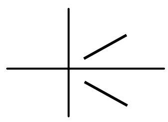




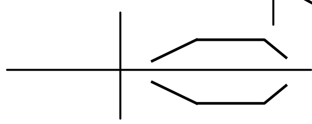
Its matrix representation is $\begin{bmatrix} \mathrm{b-a} & \\ & \mathrm{c-b} \end{bmatrix}$ which is a 2×2 matrix of 2×2 real matrices.

Similarly, the matrix representation of P_3 is of the form $\begin{bmatrix} e_0 & & \\ & e_1 & \\ & & e_2 \end{bmatrix}$ where each e_i is a 2×2 real matrix.

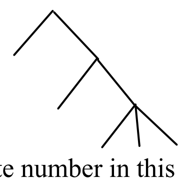
In general, $C^*(P_n)$, the C*-algebra associated to the path P_n, is isomorphic to $M_n(\boldsymbol{C})$, i.e., the module of the square matrices of dimension n over $\boldsymbol{C}$, the field of the complex numbers.
All the matrices we have dealt with so far are diagonal.
The involution in the complex plane is the transpose, which maps a complex number x+yi into its complex conjugate, x-yi, thus



For a graph with three edges, the involution acts by "twisting" the graphs, thus



With more complex graphs, such as



we proceed as follows:

For each possible path in the graph.
    index the path (they are a finite number in this case).
    and associate the corresponding C*-algebra (it will be a $M_k(\boldsymbol{C})$, for some k>0).
Then the C*-algebra associated with the graph will be the direct sum (or direct product, in our case is the same as we are dealing with n-dimensional spaces) of the C*-algebras of all the paths.

5.6.1 The Second Twist

To pass from the elicited referential to the pseudo-referential, we just apply the involution to the associated matrix. This preserves the incidence of the edges because the involution preserves topological invariants and the incidence matrix is a topological invariant. The second twist is, therefore, well defined by the involution.

5.6.2 The First Twist and the Double Composite

Consider an outcome (a vertex) in the deontic which is mapped to a dual apex in the existential, which is 'above' a set of events (edges). Then the corresponding 'element' of the composite existential-deontic articulation is an edge having as one of its vertices the outcome from the deontic and as the other vertex the set of edges of the existential.
Let v be a vertex in the deontic and let $\{e_1, e_2, \ldots, e_k\}$ be a set of edges in the existential, which could be understood as a path in a graph that has the same vertices as those in the zero-level of the existential.
This path is isomorphic to P_k which is, in turn, isomorphic to a point in a k-dimensional vector-space.
Then $\{v, P_k\}$, which can also be represented by the column vector $\begin{bmatrix} v \\ P_k \end{bmatrix}$, represents the required edge in the composite articulation.
Let V be the vector space to which this column vector belongs;
let $B = \{b_1, b_2, ..., b_{k+1}\}$ be the basis of V; and
let $B^* = \{b^1, b^2, ..., b^{k+1}\}$ be the basis of the dual space, V* (see Appendix A4);
then $v = b^1 \begin{bmatrix} v \\ P_k \end{bmatrix}$ and $e_i = b^{i+1} P_k$

6 Implications

The atoms of a local universe that emerges through the composition of existential and deontic articulations are never 'the thing itself' – actual atoms. In this sense, the 'reality' made present by these atoms approaches a beyond of 'reality' that is never known in-itself.
The demand situations in the referential that are made tractable in relation to the other two articulations through their re-constituting in the form of their involution render tractable another kind of beyond that is never known in itself, being always an anticipation of satisfaction rather than the satisfaction itself, and which occupies the same place as the unconscious in Freud's and Lacan's theorising of human being.
The significance of the triple over the double articulation is therefore just this: that human being is located between two forms of unknowability, rather than as some transcendent process of abstraction with respect to just one form of unknowability.
Those two limits of unknowability could be understood in mathematical terms as the zero and the infinity in an inversion applied on the complex unitary circle. Elements closest to the origin will be mapped into elements that are closest to the "infinity'. This 'infinity' could be understood as the centre of a circle in the projective geometry, the exterior of the complex circle being the interior of the projective circle and vice-versa. This inversion is also associated to the inversion in order described by the Galois connection as a result of the second twist.

7 Open Questions

7.1 Galois Connections

Every partially ordered set (or *poset*), *P*, gives rise to a **dual** (or **opposite**) partially ordered set, often denoted by P^{op}, which is defined to be the set with the inverse order, i.e. $x \leq y$ holds in P^{op} iff $y \leq x$ holds in *P*.

A **Galois connection**[10] is a particular correspondence between two posets.

Definition 16: Suppose $(A, \leq^A)$ and $(B, \leq^B)$ are two posets.

A *Galois connection* between these posets consists of two monotone functions:

$F : A \to B$ and $G : B \to A$, such that, for all *a* in *A* and *b* in *B*,

$F(a) \leq^B b$ if and only if $a \leq^A G(b)$.

In this situation, *F* is called the **lower adjoint** of *G* and *G* is called the **upper adjoint** of *F*.

[10] Galois connections generalize the correspondence between subgroups and subfields investigated in Galois theory.

The elicited referential articulation and the associated pseudo-referential are related through a Galois connection, the ordering on their elements being opposite.

In the following, we consider a (monotone) Galois connection $f = (f^*, f^*)$, where $f^*: A \rightarrow B$ is the lower adjoint as introduced above.

Some helpful and instructive basic properties can be obtained immediately.

By the defining property of Galois connections, $f^*(x) \leq^B f^*(x)$ is equivalent to $x \leq^A f^*(f^*(x))$, for all x in A.

Similarly, $f^*(f^*(y)) \leq^B y$, for all y in B.

These properties can be described by saying that the composite f^* o f^* is *deflationary*, while f^* o f^* is *inflationary* (or *extensive*).

Now if one considers any elements x and y of A such that $x \leq^A y$, then one can clearly use the above findings to obtain $x \leq^A f^*(f^*(y))$. Applying the basic property of Galois connections, one can now conclude that $f^*(x) \leq^B f^*(y)$. But this just shows that f^* preserves the order of any two elements, i.e. it is monotone. Again, a similar reasoning yields monotonicity of f^*. Thus monotonicity does not have to be included in the definition explicitly. However, mentioning monotonicity helps to avoid confusion about the two alternative notions of Galois connections.

Another basic property of Galois connections is the fact that $f^*(f^*(f^*(x))) = f^*(x)$, for all x in B.

Clearly we find that $f^*(x) \leq^A f^*(f^*(f^*(x)))$ because f^* o f^* is inflationary as shown above.

Similarly, since f^* o f^* is deflationary, one finds that $f^*\ f^*\ f^*\ f^*(x) \leq^B f^*\ f^*(x) \leq^B x$, which is equivalent to $f^*(f^*(f^*(x))) \leq^A f^*(x)$. This shows the desired equality.

Furthermore, we can use this property to conclude that $f^*(f^*(f^*(f^*(x)))) = f^*(f^*(x))$, i.e. f^* o f^* is *idempotent*.

The above findings can be summarized as follows: for a Galois connection, the composite f^* o f^* is monotone (being the composite of monotone functions), inflationary, and idempotent. The significance of these properties of the Galois connection on the theory of the triple articulation has yet to be investigated.

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APPENDICES

A1. Directed Graph Definitions

A directed graph is a 4-tuple, (V, E, r, s), with

V, the set of **vertices**

E, the set of **edges**

r: $E \rightarrowtail V$, the **range** of each edge: the vertex, if any, to which the edge is directed

s: $E \rightarrowtail V$, the **source** of each edge: the vertex, if any, from which the edge is directed

The **successors** of a vertex are the vertices in the range of each edge of which the vertex is the source.

succ: $V \leftrightarrow V$ succ = $r \circ s^{-1}$

The **nodes** of a directed graph are those of its vertices that have successors

nodes: $\wp V$ nodes = dom succ

The **roots** of a directed graph are those of its nodes that have no predecessors

roots: $\wp V$ roots = nodes - ran succ

The **leaves** of a directed graph are those of its vertices that have no successors

leaves: $\wp V$ leaves = V - nodes

The **body** of a directed graph comprises those of its vertices that have predecessors and successors

body: $\wp V$ body = nodes - (leaves $\cup$ roots)

The **descendants** of a vertex are its successors and their successors, transitively

desc: $V \leftrightarrow V$ desc = $succ^{+}$

The **terminal descendants** of a vertex are those of its descendants that are leaves

term: $V \leftrightarrow V$ term = desc $\cap$ (V $\times$ leaves)

The **non-terminal descendants** of a vertex are those of its descendants that are nodes

nont: $V \leftrightarrow V$ nont = desc - term

The **incident edges** of a directed graph are those edges whose source is undefined

inc: $\wp E$ inc = E - dom s

The **emergent** edges of a directed graph are those edges whose range is undefined

em: $\wp E$ em = E - dom r

The **paths** of a directed graph are those sequences of vertices, each element of which, after the first, is a successor of the previous element.

paths: $\wp$(seq V) paths = $\{p: \text{seq } V \mid \forall i: [1 .. \#p-1]. (p(i), p(i+1)) \in succ\}$

The **distance** between a vertex and one of its descendants is the length of the shortest path between them.

distance: $V \times V \rightarrowtail NAT$ $\forall(a,b)$:desc. distance(a,b) =

$\min\{n:NAT \mid \exists p:paths.\ n = \#p \wedge p(1) = a \wedge p(n) = b\}$

Subgraphs may be selected by specifying either a set of vertices or a set of edges.

Given a directed graph, G = (V, E, r, s), a set of vertices, $v \subseteq V$, selects the subgraph

vsub(v, G) = (v, E_v, r_v, s_v), where

$r_v = r \cap (E \times v)$

$s_v = s \cap (E \times v)$

$E_v = dom(r_v \cup s_v)$

and a set of edges, $e \subseteq E$, selects the subgraph

esub(e, G) = (V_e, e, r_e, s_e), where

$r_e = r \cap (e \times V)$

$s_e = s \cap (e \times V)$

$V_e = ran(r_e \cup s_e)$

A **directed acyclic graph** is directed graph, none of whose vertices is its own descendant.

desc $\cap$ Id(V) = $\varnothing$

A2. Set Theory and Logic symbols

The symbols for relational inverse ($^{-1}$), composition (°), product (×), union (∪), intersection (∩), membership (∈), null set (∅), subset (⊆), powerset (℘), cardinality (#), transitive closure ($^{+}$), image ([]), dom(ain), ran(ge), seq(uence), relation (↔), function (→) and partial function (⇸) have their usual meanings. The following are less familiar:

† **functional override**: given any two functions, f and g,

(f †g) x = g x, when x ∈ dom g

= f x, otherwise.

↑ **domain restriction**: given any relation R: X↔Y, and any set Z⊆X,

(x,y) ∈ R↑Z if and only if (x,y) ∈ R and x ∈ Z.

Id **identity relation**: given any set, X, Id(X) = {(x,x) | x∈X}

inj **injection**: given any function f: X→Y,

inj(f) if and only if ∀x,y:X. f x = f y → x = y

A3. Matrices and Complex Numbers

1) Every complex number a+bi may be uniquely represented by the point (a,b) on the Argand plane which, in turn, denotes a vector from the origin to that point which, in turn, is the sum of two orthogonal vectors, a(1,0) + b(0,1).

2) Every complex number is uniquely represented by a pair of real numbers, and vice versa, so $\boldsymbol{C} \approx \Re^2$.

3) Every complex number is also uniquely represented by a 2×2 real-valued matrix, so $\boldsymbol{C} \approx M_2(\Re) \approx \Re^2$.

For example, $a + bi = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, which suggests that we identify 1 and i, the real and the imaginary units, with $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, respectively.

The **conjugate transpose** (or **adjoint**) of an *m*-by-*n* matrix *A* with complex entries is the *n*-by-*m* matrix *A** obtained from *A* by taking the transpose and then taking the complex conjugate of each entry.

For example, if $A = \begin{bmatrix} 3+i & 2 \\ 2-2i & i \end{bmatrix}$, then $A^* = \begin{bmatrix} 3-i & 2+2i \\ 2 & -i \end{bmatrix}$ and given a vector, say, v = (1,0),

the image of v through A, Av, is (3+i, 2-2i) and through A*, A*v, is (3-i, 2).

The cojugate transpose has the following properties:

(A+B)* = A*+B* for any two matrices A and B of the same format.

(rA)* = r*A* for any complex number r and any matrix A. Here r* refers to the complex conjugate of r.

(AB)* = B*A* for any m-by-n matrix A and any n-by-p matrix B.

(A*)* = A for any matrix A.

<Ax,y> = <x, A*y> for any m-by-n matrix A, any vector x in $\boldsymbol{C}^n$ and any vector y in $\boldsymbol{C}^m$,

where <.,.> denotes the ordinary Euclidean inner product (or dot product) on $\boldsymbol{C}^m$ and $\boldsymbol{C}^n$.

This property means that * is an ***alternate tensor***. It describes the action of A* as making a symmetry plus a contraction or expansion.

The algebra $M_n(\boldsymbol{C})$ of *n*-by-*n* matrices over ***C*** becomes a C*-algebra if we consider matrices as operators on the Euclidean space $\boldsymbol{C}^n$ and use the operator *norm* (||.||) on matrices. The involution is given by the conjugate transpose.

A4. Vector Spaces and their Duals

Given any vector space V over some field F, we define the dual space V* to be the set of all linear functionals on F, i.e., scalar-valued linear-transformation on V (in this context, a "scalar" is a member of the base-field F).

V* itself becomes a vector space over F under the following definition of addition and scalar multiplication:

for all ϕ, φ in V*, a in F and x in V,

$(\phi+\varphi)(x) = \phi(x) + \varphi(x)$

$(a\phi)(x) = a\phi(x)$

In the language of tensors, elements of V are called contravariant vectors, and elements of V*, co-variant vectors, co-vectors or one-forms.

If V is finite, then V* has the same dimension as V.

Suppose $\{e_1,...,e_n\}$ is a basis for V, then the associated dual basis $\{e^1,...,e^n\}$ of V* is given by:

$$e^i(e_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

If we can interpret $\Re^n$ as the space of columns of n real numbers, its dual space is typically written as the space of rows of n real numbers. Such a row acts on $\Re^n$ as a linear functional by ordinary matrix multiplication.
If V consists of the space of geometrical vectors (arrows) in the plane, then the elements of the dual V* can be intuitively represented as collections of parallel lines. Such a collection of lines can be applied to a vector to yield a number in the following way: one counts how many of the lines the vector crosses.
If f: V $\rightarrow$ W is a linear map, we define its transpose, ${}^t f : W^* \rightarrow V^*$ by

$${}^t f(\phi) = \phi \circ f, \text{ for every } \phi \text{ in } W^*$$

The assignment $f \mapsto {}^t f$ produces an injective homomorphism between the space of linear operators from V to W and the space of linear operators from W* to V*.
That homomorphism is an isomorphism iff W is finite-dimensional.
If the linear map f is represented by the matrix A with respect to two bases of V and W, then ${}^t f$ is represented by ${}^t A$, the conjugate transpose of A with respect to the dual bases of W* and V*.
If g: W $\rightarrow$ X is another linear map, we have ${}^t(g \circ f) = {}^t f \circ {}^t g$.
In the language of category theory, taking the dual of vector spaces and the transpose of linear maps is therefore a contravariant functor from the category of vector spaces over F to itself.