The Landscape of a Simplicial Complex

P.Boxer & B. Cohen, BRL & City University

Abstract

The simplicial complex provides a useful model for a wide variety of systems, including many that are social in nature. Q-analysis computes certain measurable structural properties of simplicial complexes that correlate with interesting organisational attributes and can be used by organisations to investigate alternative development strategies. Striking visual evidence of these properties can be produced by displayed them in the form of a 3-dimensional histogram, which we call a ***landscape***. We define the simplicial complex and its properties in first order logic and provide an example landscape calculated on an arbitrary relation.

1. Introduction

The *enterprise* is a complex social system whose viability and performance depend critically on the structure of the relationships among the suppliers of, and customers for, products and services. Under the influences of globalisation and digitisation, customers expect services that are appropriate to their specific contexts-of-use, and experience *value deficit* when these expectations fail to be met. This *asymmetry* of demand forces enterprises to adopt a *relational* approach in which their strategy must be organised at the level of the particular relation to the customer's demand, rather than at the level of the enterprise as a whole, thereby increasing the variety and complexity of those relationships, and the *risks* of engaging in them [BOX1].

These risks can be exposed, and their effects anticipated, by the praxis of *triply articulated modeling* [BOX2], which elicits the enterprise's models of itself and analyses them to identify those flaws that induce the risks. Since the presentation of such flaws to the client enterprise is an essential aspect of the praxis, the quality of their visualisation is critically important to its success.

The computational tools supporting triply articulated modeling use Q-analysis, the methodology developed by Ron Atkin [ATK] for investigating the structural properties of complex systems and which has been widely applied to the analysis of social and transport systems [JOH]. Its measures of structural properties derive from the *simplicial complex*, a combinatorial model of a toplogical space [KRA].

We present some extensions to the standard operators of Q-analysis which produce a presentation of the topological structure of an enterprise's models of itself in the form of a 3-dimensional histogram, whose 'holes' and 'peaks' graphically indicate certain kinds of flaw in these models, which are symptomatic of the risks associated with adopting the relational approach**.**

2. Q-Analysis

A simplicial complex is a binary relation presented as a boolean matrix. Each element of the relation's domain labels a row of the matrix, called a *simplex*, and each element of the range, called a *vertex*, labels a column, so that each simplex is a boolean vector , the value of each element of which is True if that simplex-vertex pair is in the relation and *False* otherwise.

A simplex may be considered to be a q-dimensional surface in an n-dimensional space, where n is the number of vertices and q is one less than the number of vertices in the simplex.

Q-analysis computes values for certain topological properties of this space. For example:

- Two simplexes *share a common face* if at least one vertex is in both.
- The *dimension* of this common face is one less than the number of shared vertices.
- A simplex is *q-connected* to all those simplexes, other than itself, with which it shares a common face of dimension greater than or equal to q.
- A *q-component* of a simplicial complex is a maximal set of its simplices that are q-connected to each other (transitively).
- The *structure vector* of a simplicial complex gives the number of q-components at each dimension, q.

Our extension of the standard Q-analysis operators is based on the definition of an *ordering* on the simplices of a simplicial complex which is such that, by erecting at each simplex a column of height equal to its greatest q-connectivity, we present a histogram in which the elements of each q-component are adjacent to each other. The envelope of this histogram is a curve whose maxima and minima (*peaks* and *valleys*) graphically indicate the areas of strongest and weakest simplex connectivity.

We further define a concept of *connection strength* which adds a further dimension to the histogram. This 3-dimensional structure is called a *landscape*.

In the next section, we present formal definitions of these, and other, properties, together with the values they take in an example simplicial complex, which was constructed from the analysis of a real enterprise.

3. Formal Definitions

An illustrative simplicial complex, generated from a model of the aerospace industry, is shown in Figure 1. It represents a company's model of the supply structure of its industry, with companies on the rows (simplexes) and their capabilities in the columns (vertices). Each measure defined in the text is exemplified by calculating its value on this simplcial complex.

Given any *binary relation*, $R:X \leftrightarrow Y$, each element of Y is called a *vertex* and each element, s, of X names the set of vertices, R[{s}] (the image of s through R), which is called a *simplex*. The set of simplexes that define the entire relation (i.e. the *corresponding function* of the relation) form a *simplicial complex*. A simplicial complex may be presented as a Boolean matrix whose rows are the simplexes and whose columns are labelled by the vertices. The example has 34 simplexes and 9 vertices, the value *True* being represented by a '1' and *False* by a blank. Thus, the ***simplicial complex*** of a relation, $R: X \leftrightarrow Y$, is defined by:

$$\mathrm{sc}: (X \leftrightarrow Y) \rightarrow (X \times Y \rightarrow \mathrm{Bool})$$
$$\forall R:X \leftrightarrow Y;\ s:X,\ v:Y \bullet \mathrm{sc}\ R\ (x,y) \leftrightarrow xRy$$

Note that the simplicial complex is invariant with respect to reordering of both simplexes and vertices. We will exploit that to choose an ordering of simplexes that makes the output display aesthetically pleasing.

The ***upper bound*** of a simplex in R is *one less than* its cardinality.

$$\mathrm{ub}: (X \leftrightarrow Y) \rightarrow X \rightarrow \mathbf{N}$$
$$\forall R:X \leftrightarrow Y;\ s:X \bullet \mathrm{ub}\ R\ s = \#R[\{s\}] - 1$$

The upper bound of p_allied_signal is 7.

The ***common face*** of a pair of simplexes in R is the set of vertices that they share. Pairs of simplexes that share no vertices are *disjoint*.

$$\mathrm{cface}: (X \leftrightarrow Y) \rightarrow (X \times X) \rightarrow \wp Y$$
$$\forall R:X \leftrightarrow Y;\ s,t:X \bullet \mathrm{cface}\ R\ (s,t) = R[\{s\}] \cap R[\{t\}]$$

The common face of p_allied_signal and p_rolls_royce is the set (r&o_capability, system_intgn_capability, control_system_expertise, engine_mfr}.

Two distinct simplexes in R are said to be ***adjacent*** if they have a non-empty common face.

$$\mathrm{adj}: (X \leftrightarrow Y) \rightarrow (X \leftrightarrow X)$$
$$\forall R:X \leftrightarrow Y;\ s,t:X \bullet$$
$$(s,t) \in \mathrm{adj}\ R \Leftrightarrow s \neq t \wedge \mathrm{cface}\ R\ (s,t) \neq \emptyset.$$

[Note that this relation is *symmetric*.]
p_allied_signal and p_rolls_royce are adjacent.

An ***isolated*** simplex in R has no adjacent simplex.

$$\mathrm{iso}: (X \leftrightarrow Y) \rightarrow \wp X$$
$$\mathrm{iso}\ R = X - \mathrm{dom}\ \mathrm{adj}\ R$$

No simplex in the example is isolated.

Figure 1: An Example of a Simplicial Complex

	air_oil_separator_mfr	r&o_capability	airframe_hx_mfr	engine_hx_mfr	system_intgn_capability	control_system_expertise	actuation_mfr	valve_mfr	engine_mfr
allied_signal					1	1	1	1	1
lori		1	1	1					
garret			1	1					
ngl			1	1					
secan			1	1					
airmark		1							
p&w		1			1	1			1
hamilton_standard		1	1		1	1		1	
rolls_royce	1	1			1	1			1
allison_engine_co					1	1			1
behr			1	1		1	1		
grimes		1	1	1		1	1	1	
hughes_treitler			1	1					
marston		1	1	1			1		
micro_turbo		1	1	1			1		
sumitomo			1	1					
serck		1	1	1					
stewart_warner		1	1	1					
dunlop_equipment	1						1	1	
uap		1					1	1	
lucas_varity		1			1	1	1	1	
woodward_governer								1	
ge					1	1			1
snecma					1	1			1
cfm	1				1	1			1
brr					1	1			1
iae					1	1			1
textron_lycoming					1	1			1
sumitomo_hamilton_standard		1	1	1	1	1		1	
brr_iae_rolls_royce	1	1			1	1			1
p_allied_signal		1	1	1	1	1	1	1	1
p_utc		1	1		1	1		1	1
p_rolls_royce	1	1			1	1			1
p_btr	1	1	1	1			1	1	

The ***lower bound*** of a simplex in R is the dimension (one less than the cardinality) of the largest common face (if any) that it shares with any simplex in R *other than itself*.

$$\mathrm{lb}: (X \leftrightarrow Y) \rightarrow X \ \rightarrowtail \mathbf{N}$$
$$\forall R:X \leftrightarrow Y;\ s:X \bullet s \notin \mathrm{iso}\ R \Rightarrow$$
$$\mathrm{lb}\ R\ s = \max\{n:\mathbf{N} \mid \exists t:X \bullet \mathrm{cface}\ R\ (s,t) = n\} - 1$$

The lower bound of p_allied_signal is 5.

The ***eccentricity*** of a simplex in R is the difference between its upper and lower bounds divided by *one more than* its lower bound. The eccentricity of an isolated simplex is not defined.

$$\text{ecc}: (X \leftrightarrow Y) \rightarrow X \nrightarrow \mathbf{Q}$$
$$\forall R:X \leftrightarrow Y;\ s:X \bullet\ s \notin \text{iso } R \Rightarrow$$
$$\text{ecc } R\ s = (\text{ub } R\ s - \text{lb } R\ s)/(\text{lb } R\ s + 1)$$

The eccentricity of p_allied_signal is (7-5)/6 = 0.33

A simplex in R is ***q-connected***, for some integer q, to all those simplexes in R, other than itself, with which it shares a common face of dimension *at least* q (i.e. of cardinality *at least* q-1).

$$\text{qconn}: (X \leftrightarrow Y) \rightarrow \mathbf{N} \rightarrow (X \leftrightarrow X)$$
$$\forall R:X \leftrightarrow Y;\ q:\mathbf{N};s,t:X \bullet\ (s,t) \in \text{qconn } R\ q \Leftrightarrow$$
$$(s,t) \in \text{adj } R \wedge \#(\text{cface } R\ (s,t)) \geq q-1$$

[Note that this relation is *symmetric* and *irreflexive*.] p_rolls_royce is 3-connected to p_utc, p_allied_signal, brr_iae_rolls_royce, cfm, rolls_royce and hamilton_standard.

A simplex in R is ***transitively q-connected***, at some dimension q, to all those simplexes in R to which it is q-connected and to all those to which they are, in turn, q-connected.

$$\text{tqconn}: (X \leftrightarrow Y) \rightarrow \mathbf{N} \rightarrow (X \leftrightarrow X)$$
$$\forall R:X \leftrightarrow Y;\ q:\mathbf{N};s,t:X \bullet\ (s,t) \in \text{tqconn } R\ q \Leftrightarrow$$
$$(s,t) \in \text{qconn } R\ q \vee$$
$$\exists c:X \bullet ((s,c) \in \text{tqconn } R\ q \wedge (c,t) \in \text{tqconn } R\ q)$$

[Note that this is an *equivalence relation*.] p_rolls_royce is transitively 3-connected to p_btr, p_utc, p_allied_signal, brr_iae_rolls_royce, sumitomo-hamilton_standard, cfm, lucas_varity, grimes, behr, rolls_royce, hamilton_standard, p&w and alllied_signal.

A ***q-component*** is a *non-singular* maximal set of simplexes in R that are transitively q-connected to each other, i.e. the q-components are the *non-singular partitions* of the set of simplexes induced by the transitive q-connectedness equivalence relation.

$$\text{qcomps}: (X \leftrightarrow Y) \rightarrow (\mathbf{N} \leftrightarrow \wp X)$$
$$\forall R:X \leftrightarrow Y;\ q:\mathbf{N};ss:\wp X \bullet\ ss \in \text{qcomps } R\ q \Leftrightarrow$$
$$\#ss > 1 \wedge \forall s,t:ss \bullet\ (s,t) \in \text{tqconn } R\ q$$

4. Simplex Ordering

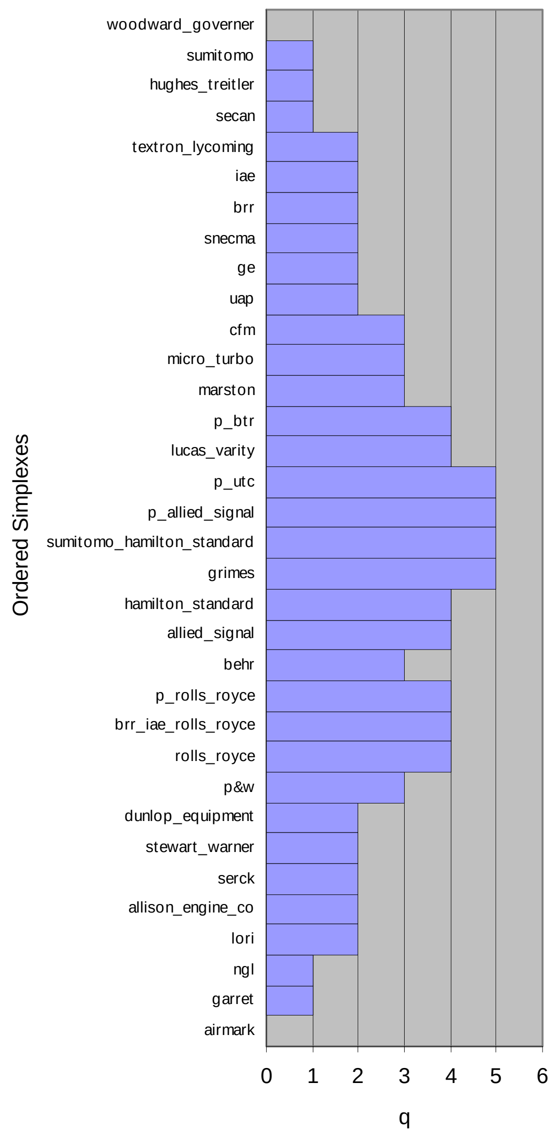
A simplicial complex can be presented as a histogram in which the simplexes are listed on the horizontal axis and their q-connectivity is plotted on the vertical axis. If the simplexes are appropriately ordered along the horizontal axis, then the q-components will appear as horizontal bands in this histogram and the peaks will appear as maxima of the histogram's envelope.

An appropriate ***simplex ordering*** is a sequence of the simplexes of a simplicial complex in which each simplex appears exactly once and the simplexes of every q-component are contiguous.

$$\text{ord}: (X \leftrightarrow Y) \rightarrow \text{seq } X$$
$$\forall R:X \leftrightarrow Y \bullet\ \#(\text{ord } R) = \#(\text{dom } R)$$
$$\wedge \text{ ran ord } R = \text{dom } R$$
$$\wedge\ \forall q:\text{dom qcomps } R \bullet\ \forall ss:\text{qcomps } R\ q \bullet$$
$$\exists i,j:\mathbf{N}:\ ss = (\text{ord } R)[i..j]$$

Although this definition does not specify a unique simplex ordering, an acceptable one is always computable. Such an algorithm is implemented as a functional program that, when applied to our example simplicial complex, generates the histogram shown in figure 2.

Figure 2



The ***structure vector*** of R is the number of q-components at each dimension q.

$$sv: (X \leftrightarrow Y) \to \mathbf{N} \to \mathbf{N}$$
$$\forall R:X \leftrightarrow Y;\ q:\mathbf{N} \bullet\ sv\ R\ q = \#((qcomps\ R)\ [\{q\}])$$

The structure vector of our example simplicial complex is graphically displayed in figure 2, which shows that there is one component at every value of q except for q=3, where there are two components.

A ***peak*** of R is a q-component none of whose subsets is a q-component of dimension higher than it.

$$peaks: (X \leftrightarrow Y) \to \wp\,\wp X$$
$$\forall R:X \leftrightarrow Y \bullet\ peaks\ R =$$
$$\{ss: \wp X \mid \exists q:\mathbf{N} \bullet\ (q, ss) \in qcomps\ R$$
$$\wedge \neg(\exists bs: \wp ss;\ r:\mathbf{N} \bullet\ (r, bs) \in qcomps\ R \wedge r>q)\}$$

Our example has two peaks, as is clear from figure 2.

The ***structural upper bound*** of a peak in R is the highest dimension, q, at which its simplexes are q-connected.

$$pub: (X \leftrightarrow Y) \to \wp\,\wp X \mathrel{+\!\!\!\to} \mathbf{N}$$
$$\forall R:X \leftrightarrow Y;\ ss: peaks\ R;\ q:\mathbf{N} \bullet\ pub\ R\ \ ss =$$
$$\max\{q:\mathbf{N} \mid (q,ss) \in qcomps\ R)\}$$

The ***structural lower bound*** of a peak in R is the highest dimension, q, at which any of its simplexes is q-connected to a simplex of a different peak, or –1 if it is isolated.

$$plb: (X \leftrightarrow Y) \to \wp\,\wp X \mathrel{+\!\!\!\to} \mathbf{Z}$$
$$\forall R:X \leftrightarrow Y;\ ss: peaks\ R \bullet\ plb\ R\ ss =$$
$$\max(\{n:\mathbf{N} \mid \exists bs:peaks\ R \bullet\ ss \neq bs \wedge$$
$$((ss \times bs) \cap (qconn\ R\ n)) \neq \emptyset\} \cup \{-1\})$$

The ***granularity*** of a peak in R is the difference between its structural upper and lower bounds, divided by its structural lower bound plus one.

$$gran: (X \leftrightarrow Y) \to \wp\,\wp X \to \mathbf{Q}$$
$$\forall\ R:X \leftrightarrow Y;\ ss: peaks\ R \bullet\ gran\ R\ ss =$$
$$((pub\ R\ ss) - (plb\ R\ ss))\ /\ ((plb\ R\ ss) + 1)$$

Several properties of our example simplicial complex are tabulated against the ordered list of simplexes in Figure 3.

We may further enrich the visualisation of simplicial complexes by considering the set of q-connections in which each simplex participates.

The ***q-set*** of a simplex in R is the *set* of other simplexes with which it is q-connected.

$$qset: (X \leftrightarrow Y) \to X \to \mathbf{N} \to \wp X$$
$$\forall R:X \leftrightarrow Y;\ s:X;\ q:\mathbf{N} \bullet$$
$$qset\ R\ s\ q = \{t:X \mid (s,t) \in qconn\ R\ q\}$$

The cardinality of this set measures the *strength* of a simplex's q-connectivity, which we use to refine our definition of *component*, as follows:

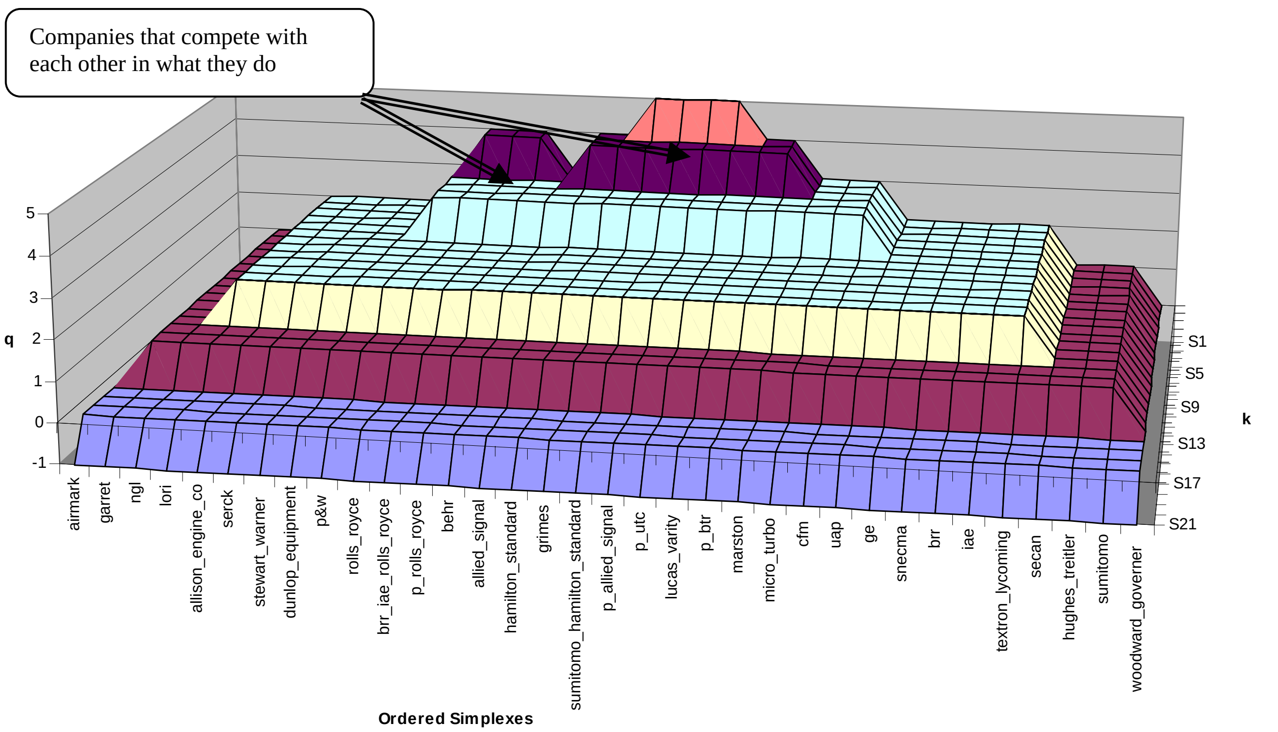
Figure 3	UpperBound	LowerBound	Eccentricity	StructUB	StructLB	Granularity
airmark	0	0	0	4	4	0
garret	1	1	0	4	4	0
ngl	1	1	0	4	4	0
lori	2	2	0	4	4	0
allison_engine_co	2	2	0	4	4	0
serck	2	2	0	4	4	0
stewart_warner	2	2	0	4	4	0
dunlop_equipment	2	2	0	4	4	0
p&w	3	3	0	4	4	0
rolls_royce	4	4	0	4	4	0
brr_iae_rolls_royce	4	4	0	4	4	0
p_rolls_royce	4	4	0	4	4	0
behr	3	3	0	5	3	0.5
allied_signal	4	4	0	5	3	0.5
hamilton_standard	4	4	0	5	3	0.5
grimes	5	5	0	5	3	0.5
sumitomo_hamilton_standard	5	5	0	5	3	0.5
p_allied_signal	7	5	0.33	5	3	0.5
p_utc	5	5	0	5	3	0.5
lucas_varity	4	4	0	5	3	0.5
p_btr	5	4	0.2	5	3	0.5
marston	3	3	0	5	3	0.5
micro_turbo	3	3	0	5	3	0.5
cfm	3	3	0	5	3	0.5
uap	2	2	0	5	3	0.5
ge	2	2	0	5	3	0.5
snecma	2	2	0	5	3	0.5
brr	2	2	0	5	3	0.5
iae	2	2	0	5	3	0.5
textron_lycoming	2	2	0	5	3	0.5
secan	1	1	0	5	3	0.5
hughes_treitler	1	1	0	5	3	0.5
sumitomo	1	1	0	5	3	0.5
woodward_governer	0	0	0	5	3	0.5

A ***kq-component*** of R is a maximal set of simplexes each of which is q-connected to at least *k* other simplexes *in that set.*

$$kqcomps: (X \leftrightarrow Y) \to \mathbf{N} \to (\mathbf{N} \leftrightarrow \wp X)$$
$$\forall R:X \leftrightarrow Y;\ k,q:\mathbf{N};ss: \wp X \bullet$$
$$(q, ss) \in kqcomps\ R\ k \Leftrightarrow$$
$$\forall s:ss \bullet\ \#((qset\ R\ s\ q) \cap ss) \geq k$$

The 1q-components of a simplicial complex are just its q-components, since each of the simplexes of a q-component is q-connected to at least one of the others. The graph of q-components may therefore be considered as the first of a series of planes, indexed by *k*, in which the kq-components are plotted using the same simplex ordering, as shown in figure 4.

Figure 4



5. Landscapes

This presentation is a ***landscape***: a three-dimensional surface pocked with holes. Since, by definition, every kq-component must contain at least k+1 simplexes, every component in the 1-plane of a landscape must have at least 2 simplexes (i.e. non-singular, as we said before); every component in the 2-plane must have at least 3 simplexes; and so on.

At each value of *k*, each simplex in R supports a column, whose ***height*** is the greatest value of *q* at which the simplex is in a kq-component.

ht: $(X \leftrightarrow Y) \rightarrow \mathbf{N}^{+} \rightarrow X \rightarrow \mathbf{N}$ $\forall R:X \leftrightarrow Y;\ k:\mathbf{N}^{+};\ s:X\bullet$ ht R k s = max{q:**N** | $\exists ss: \wp X\bullet\ s \in ss \wedge (q, ss) \in$ kqcomps R k}

In the 1-plane, the height of each simplex's column is its lower bound. As well as the k-planes, we can construct another plane with the same simplex ordering in which each simplex supports a column whose height is equal to its upper bound. The proportional reduction in height of each simplex's column in passing from this '0-plane' to the 1-plane is equal to the simplex's eccentricity.

The ***k-eccentricity*** of a simplex in R is the proportional reduction in height of the simplex's column in passing from any k-plane to the (k+1)-plane. Under this definition, eccentricity becomes 0-eccentricity.

kecc: $(X \leftrightarrow Y) \rightarrow \mathbf{N} \rightarrow X \rightarrow \mathbf{Q}$ $\forall R:X \leftrightarrow Y;\ k:\mathbf{N}\bullet$ (k=0 $\wedge$ kecc R k = ecc R) $\vee$ (k>0 $\wedge$ $\forall s:X\bullet$ kecc R k s = ((ht R k s) – (ht R (k+1) s))/ ((ht R (k+1) s) +1)

Further structure in a relation may be made visible by displaying the simplicial complexes of its complement and of its inverse.

The ***negative*** simplicial complex of a relation, R, is the corresponding function of the complement of R:

negsc: $(X \leftrightarrow Y) \rightarrow (X\times Y \rightarrow$ Bool) $\forall R:X \leftrightarrow Y;\ s:X,\ v:Y\bullet$ negsc R (x,y) $\Leftrightarrow \neg$xRy

To negate the example, replace each '1' with a blank and vice versa.

The ***transposed*** simplicial complex of a relation, R, is the corresponding function of the inverse of R:

transc: $(X \leftrightarrow Y) \rightarrow (X\times Y \rightarrow$ Bool) $\forall R:X \leftrightarrow Y;\ s:X,\ v:Y\bullet$ transc R (x,y) $\Leftrightarrow$ yRx

To transpose the example, rotate it through 90°, so that the vertices become simplexes and vice versa.

The transposed simplicial complex may itself be negated. Negation and transposition are commutative.

The landscapes of the simplicial complexes generated by negating, transposing and negating the

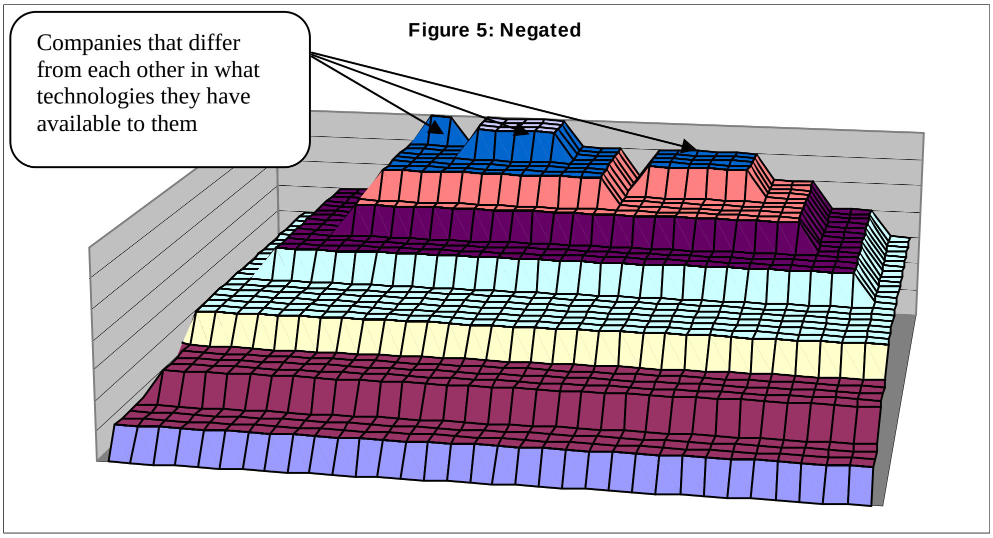


Figure 5: Negated

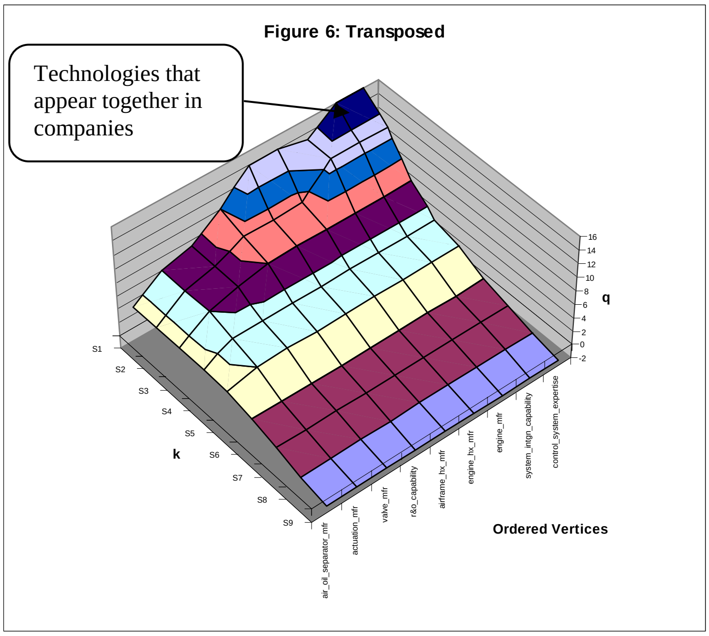


Figure 6: Transposed

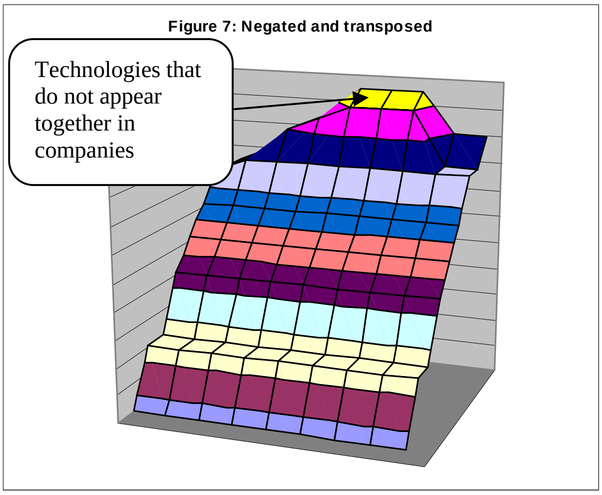


Figure 7: Negated and transposed

transpose of our example are shown in figures 5, 6 and 7. Note that in the transposed landscapes, the X-axis is labelled by the ordered vertices of the original simplicial complex. The axis labels have been omitted from the negated landscapes to minimise clutter.

6. Conclusion

The distribution of *non*-connectedness in a complex indicates the existence of 'holes', which are peaks in the *negative* of a simplicial complex. The number and granularity of peaks and holes in a simplicial complex are indicators of the complex's 'jaggedness'. For example, in a simplicial complex characterising different forms of demand (the simplexes) for products and services (the vertices), jaggedness would signify the degree of structural heterogeneity in the way that demands were made, while holes would signify their lack of interconnectedness (i.e. jaggedness is to holes as differentiation is to lack of integration). Holes are therefore important indicators of *risk*, the significance/consequence of which depends on the context of the hole.

The paper has shown an extension to Q-analysis which supports the analysis and visualisation of these holes.

References

[ATK]
[BOX1] Service Supply and Asymmetric Demand
[BOX2] The Analysis of Risk in Enterprises
[JOH]
[KRA]